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On the Neumann-Dirichlet problem for a system of nonlinear viscoelastic equations of Kirchhoff type with Balakrishnan-Taylor term

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ABSTRACT

In this paper, the Neumann-Dirichlet boundary problem for a system of nonlinear viscoelastic equations of Kirchhoff type with Balakrishnan-Taylor term is considered. At first, a local existence is established by the linear approximation together with the Faedo-Galerkin method. Then, by establishing several reasonable conditions and suitable energy inequalities, the solution of the problem admits a general decay in time.

Keywords: *System of viscoelastic equations, Kirchhoff type; Balakrishnan-Taylor term; Faedo-Galerkin method; General decay*

1 Introduction

In this paper, we consider the initial-boundary value problem for a system of viscoelastic equations of Kirchhoff type with Balakrishnan-Taylor damping as follows

$$\begin{cases} u_{tt} - \lambda u_{xxt} - \left[\mu_* + \mu_1\left(\langle u_x(t), u_{xt}(t)\rangle\right)\right] u_{xx} + \lambda_1 u_t + \displaystyle\int_0^t g_1(t-s)\, u_{xx}(s)\, ds \\ \qquad\qquad\qquad\qquad = f_1(u,v) + F_1(x,t),\ 0 < x < 1,\ t > 0, \\ v_{tt} - \mu_2\left(\|v_x(t)\|^2\right) v_{xx} + \lambda_2 v_t + \displaystyle\int_0^t g_2(t-s)\, v_{xx}(s)\, ds \\ \qquad\qquad\qquad\qquad = f_2(u,v) + F_2(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = v_x(0,t) = v(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x),\ v(x,0) = \tilde{v}_0(x),\ v_t(x,0) = \tilde{v}_1(x), \end{cases} \tag{1.1}$$

where $\lambda,\ \lambda_1,\ \lambda_2,\ \mu_*$ are given positive constants and $\tilde{u}_0,\ \tilde{v}_0,\ \tilde{u}_1,\ \tilde{v}_1,\ \mu_i,\ f_i,\ g_i,\ (i = 1,2)$ are given functions satisfying some conditions specified later. In (1.1), the nonlinear terms $\mu_1\left(\langle u_x(t), u_{xt}(t)\rangle\right)$ and $\mu_2\left(\|v_x(t)\|^2\right)$ depend on the integrals

$$\langle u_x(t), u_{xt}(t)\rangle = \int_0^1 u_x(x,t)\, u_{xt}(x,t)\, dx$$

named the Balakrishnan-Taylor damped term and $\|v_x(t)\|^2 = \int_0^1 v_x^2(x,t)\,dx$ respectively.

It is well known that the mathematical models of Kirchhoff-type equations come from describing small vibrations of an elastic stretched string. The original equation is first investigated by Kirchhoff [16] and modelled in the form

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L u_x^2(y,t)\,dy\right)u_{xx}, \tag{1.2}$$

where $u = u(x;t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension. Until now, numerous problems of Kirchhoff-type wave equations associated with various boundary and initial conditions have been studied extensively, see [5], [7], [18], [19], [24], [28], and the references therein.

The system (1.1) is regarded as a combination of the Kirchhoff-type wave equation and the equation with the Balakrishnan-Taylor damping $\langle u_x(t), u_{xt}(t)\rangle$, in which the original equation of $(1.1)_1$ was first proposed by Balakrishnan and Taylor in 1989, see [1], modelling for flight structures with viscous and nonlinear nonlocal damping in one-dimensional case

$$\begin{aligned}&\varrho u_{tt} + EIu_{xxxx} - cu_{xxt}\\ &\quad - \left[H + \frac{EA}{2L}\int_0^L |u_x|^2\,dx + \tau\left|\int_0^L u_x u_{xt}dx\right|^{2(N+\eta)}\int_0^L u_x u_{xt}dx\right]u_{xx} = 0,\end{aligned} \tag{1.3}$$

where $u = u(x,t)$ represents the transversal deflection of an extensible beam of length $2L > 0$ in the rest position, $\varrho > 0$ is the mass density, E is Young's modulus of elasticity, I is the cross-sectional moment of inertia, H is the axial force (either traction or compression), A is the cross-sectional area, $c > 0$ is the coefficient of viscous damping, $\tau > 0$ is the Balakrishnan-Taylor damping coefficient, $0 \le \tau < 1$, $0 \le \eta < \frac{1}{2}$ and $N \in \mathbb{N}$. The equation (1.3) seems to be related to the panel flutter equation and spillover problem which studied by Bass and Zes in [2]. In recent years, the equations with the Balakrishnan-Taylor damping have been received a large amount of interest, in which properties of solutions such as stability, decay and blow-up in finite time have been considered, see [3], [4], [8], [11]- [15], [17], [22], [24]- [27] and the references therein.

An important question of asymptotic behavior of solutions was raised by Clark [6] that the solutions of the proposed problem with a damping in the form $\Delta^2 u_t$ were exponentially decayed when the time t went to infinity. In [25], Tatar and Zarai considered the initial-boundary value problem

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1\|\nabla u(t)\|^2 + \xi_2\langle\nabla u(t), \nabla u_t(t)\rangle\right)\Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ \qquad\qquad\qquad\qquad = |u|^p u, \text{ in } \Omega\times(0,+\infty), \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x),\, x\in\Omega, \\ u(x,t) = 0, t > 0, x \in \partial\Omega \end{cases} \tag{1.4}$$

and proved a result of exponentially decayed energy of the solutions provided by the fact that the kernel was decayed exponentially. After that, the results in [25] have been improved in [29], in which the authors have proved a polynomial decay provided by the relaxation function g decaying polynomially and satisfying the condition $g'(t) \le -\xi(g(t))^{1+\frac{1}{p}}$ (ξ is a positive constant). In [27], Tavares et al. used the theory of C_0-semigroup to study the well-posedness and long-time dynamics of a class of extensible beams with nonlocal Balakrishnan-Taylor and frictional damping

$$u_{tt}+\Delta^2 u-\left[\beta+\gamma\left\|\nabla u\right\|_2^2+\delta\left|\langle\nabla u(t),\nabla u_t(t)\rangle\right|^{q-2}\langle\nabla u(t),\nabla u_t(t)\rangle\right]\Delta u+\kappa u_t+f(u)=h, \tag{1.5}$$

where $(x,t)\in\Omega\times\mathbb{R}_+$, and boundary conditions Dirichlet-Newman. In the case that Balakrishnan-Taylor terms appear in both hand-side of the equation, Ngoc et. al. [23] investigated the following strongly damped wave equation

$$\begin{aligned} &u_{tt}-\lambda u_{xxt}-\mu\left(t,\langle u_x(t),u_{xt}(t)\rangle,\left\|u(t)\right\|^2,\left\|u_x(t)\right\|^2\right)u_{xx}\\ &\qquad=f(x,t,u,u_x,u_t,\langle u_x(t),u_{xt}(t)\rangle,\left\|u(t)\right\|^2,\left\|u_x(t)\right\|^2),\ \ 0<x<1,0<t<T. \end{aligned} \tag{1.6}$$

By using the linear approximation combined with the Faedo-Galerkin method and the weak compact method, the authors proved the unique local existence of weak solutions. In addition, in the case $\mu=B\left(\left\|u_x(t)\right\|^2\right)+\sigma\left(\langle u_x(t),u_{xt}(t)\rangle\right)$ and $f=-\lambda_1 u_t+f\left(u\right)+F\left(x,t\right)$, they put several suitable hypotheses and sufficient conditions for the nonlinear function $\sigma(\cdot,\cdot)$ of Balakrishnan-Taylor damping to obtain an exponential decay of solutions.

Although, there have been numerous published results of single equations with Balakrishnan-Taylor damping, studies on system of equations with Balakrishnan-Taylor damping have recieved little attention. It seems that the first result of system of equations with Balakrishnan-Taylor damping has been considered by Mu and Ma [22], of which the proposed model has described as follows

$$\begin{cases} u_{tt}-\left(a+b\left\|\nabla u(t)\right\|^2+\sigma\displaystyle\int_\Omega\nabla u\nabla u_t dx\right)\Delta u+\displaystyle\int_0^t g_1(t-s)\Delta u(s)ds=f_1(u,v),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times\mathbb{R}_+,\\ v_{tt}-\left(a+b\left\|\nabla v(t)\right\|^2+\sigma\displaystyle\int_\Omega\nabla v\nabla v_t dx\right)\Delta v+\displaystyle\int_0^t g_2(t-s)\Delta v(s)ds=f_2(u,v),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times\mathbb{R}_+, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^n$; a, b, σ are given the positive constants. By the energy method, the authors obtained an arbitrary decay of solutions according to the relaxation functions g_i satisfying $g_i'\left(t\right)\le-\xi\left(t\right)g_i\left(t\right),\left(i=1,2\right)$, where $\xi\left(t\right)$ is a positive and non-increasing function. Recently, Nam et al. [20] have considered a system of asymmetric wave equations with Kirchhoff-Carrier and Balakrishnan-Taylor terms, namely

$$\begin{cases} u_{tt}-\lambda u_{xxt}-\mu_1\left(t,\langle u_x\left(t\right),u_{xt}\left(t\right)\rangle\right)u_{xx}=f_1\left(x,t,u,v,u_x,v_x,u_t,v_t\right),\\ v_{tt}-\mu_2\left(t,\left\|v\left(t\right)\right\|^2,\left\|v_x\left(t\right)\right\|^2\right)v_{xx}=f_2\left(x,t,u,v,u_x,v_x,u_t,v_t\right),\\ (x,t)\in(0,1)\times(0,T), \end{cases} \tag{1.8}$$

associated with Robin-Dirichlet boundary conditions. The authors have proved the existence and uniqueness of local solutions established by the Faedo-Galerkin method and the arguments of compactness. Furthermore, the exponential decay of solutions has been also studied. However, the general decay of solutions of (1.8) has not been studied.

In light of the aforementioned works, we inherit the methods and techniques used for [20] and [23] to establish the existence and uniqueness of local weak solutions of the problem (1.1). In additon, for certain class of relaxation functions and certain initial data, we prove that the decay rate of the solution energy is similar to that of relaxation functions which is not necessarily of exponential or polynomial type. Our paper is organized as follows. In Sect. 2, the existence and uniqueness of local weak solutions of the problem (1.1) are established. In Sect. 3, by adopting and modifying the method of [20] and [23], we show the statement and the proof of our general decay result.

2 Main results

2.1 The existence and uniqueness theorem

In this section, we shall study the existence and uniqueness of weak solution for the problem (1.1).

Definition 2.1. *The weak solution of the problem (1.1) is a pair of functions* (u,v) *belonging to the following functional spaces*

$$\big\{(u,v)\in L^{\infty}(0,T;\big(H_0^1\cap H^2\big)\times\big(V\cap H^2\big)):(u',v')\in L^{\infty}(0,T;\big(H_0^1\cap H^2\big)\times V),\\ (u'',v'')\in\big[L^2(0,T;H_0^1)\cap L^{\infty}(0,T;L^2)\big]\times L^{\infty}(0,T;L^2)\big\},$$

and satisfying the following variational problem

$$\begin{cases} \langle u''(t),\phi\rangle+\lambda\langle u_x'(t),\phi_x\rangle+\bar{\mu}_1[u](t)\langle u_x(t),\phi_x\rangle+\lambda_1\langle u'(t),\phi\rangle \\ \qquad =\int_0^t g_1(t-s)\langle u_x(s),\phi_x\rangle ds+\langle f_1[u,v](t)+F_1(t),\phi\rangle, \\ \langle v''(t),\tilde{\phi}\rangle+\mu[v](t)\left\langle v_x(t),\tilde{\phi}_x\right\rangle+\lambda_2\left\langle v'(t),\tilde{\phi}\right\rangle \\ \qquad =\int_0^t g_2(t-s)\langle v_x(s),\tilde{\phi}_x\rangle ds+\langle f_2[u,v](t)+F_2(t),\tilde{\phi}\rangle, \end{cases} \tag{2.1}$$

for all $(\phi,\tilde{\phi})\in H_0^1\times V$*, together with the initial conditions*

$$(u(0),u'(0))=(\tilde{u}_0,\tilde{u}_1),\ \ (v(0),v'(0))=(\tilde{v}_0,\tilde{v}_1), \tag{2.2}$$

where $V=\{v\in H^1:v(1)=0\}$ *and*

$$\begin{cases} \bar{\mu}_1[u](t)=\mu_*+\mu_1(\langle u_x(t),u_x'(t)\rangle),\ \mu_2[v](t)=\mu_2\left(\|v_x(t)\|^2\right), \\ f_i[u,v](x,t)=f_i(u(x,t),v(x,t)),\ i=1,2. \end{cases} \tag{2.3}$$

We make the following assumptions:

($\mathbf{H}_1$) $(\tilde{u}_0,\tilde{v}_0)\in(H_0^1\cap H^2)\times(V\cap H^2)$, $(\tilde{u}_1,\tilde{v}_1)\in(H_0^1\cap H^2)\times V$ and $\tilde{v}_{0x}=0$;
($\mathbf{H}_2$) $\mu_1\in C^1(\mathbb{R})$ and there exist a positive constant $\mu_{1*}<\mu_*$ such that $\mu_1(y)\geq-\mu_{1*},\forall y\in\mathbb{R}$;
($\mathbf{H}_3$) $\mu_2\in C^1(\mathbb{R}_+)$ and there exist $\mu_{2*}>0$ such that $\mu_2(z)\geq\mu_{2*},\forall z\in\mathbb{R}_+$;
($\mathbf{H}_4$) $g_i\in H^1(\mathbb{R}_+)$, $i=1,2$;
($\mathbf{H}_5$) $f_i\in C^1(\mathbb{R}^2)$, $F_i\in C^1([0,1]\times\mathbb{R}_+)$ $i=1,2$.

For each $T>0$, we denote

$$V_T=\{(u,v)\in L^{\infty}(0,T;(H_0^1\cap H^2)\times(V\cap H^2)):\\ (u',v')\in L^{\infty}(0,T;(H_0^1\cap H^2)\times V),(u'',v'')\in L^2(0,T;H_0^1\times L^2)\}, \tag{2.4}$$

is a Banach space with respect to the norm

$$\|(u,v)\|_{V_T}=\max\Big\{\|(u,v)\|_{L^{\infty}\left(0,T;\left(H_0^1\cap H^2\right)\times(V\cap H^2)\right)},\\ \|(u',v')\|_{L^{\infty}\left(0,T;\left(H_0^1\cap H^2\right)\times V\right)},\|(u'',v'')\|_{L^2\left(0,T;H_0^1\times L^2\right)}\Big\}. \tag{2.5}$$

We note that

$$W_1(T)=\left\{(u,v)\in C^0\big([0,T];H_0^1\times V\big)\cap C^1\left([0,T];L^2\times L^2\right):u'\in L^2\left(0,T;H_0^1\right)\right\} \tag{2.6}$$

is also a Banach space with respect to the norm (Lions [21])

$$\begin{aligned}\|(u,v)\|_{W_1(T)} = \|u\|_{C^0\left([0,T];H_0^1\right)} + \|v\|_{C^0([0,T];V)} + \|u'\|_{C^0([0,T];L^2)} \\ + \|v'\|_{C^0([0,T];L^2)} + \|u'\|_{L^2(0,T;H_0^1)}.\end{aligned} \tag{2.7}$$

For every $M>0$, we put

$$\begin{aligned}&W(M,T)=\left\{(u,v)\in V_T:\|(u,v)\|_{V_T}\le M\right\},\\ &W_1(M,T)=\left\{\ (u,v)\in W(M,T):(u'',v'')\in L^\infty\left(0,T;L^2\times L^2\right)\right\}.\end{aligned} \tag{2.8}$$

Now, we establish the following recurrent sequence $\{(u_m,v_m)\}_{m\in\mathbb{N}}$. The first term is chosen as $(u_0,v_0)\equiv(0,0)$, suppose that

$$(u_{m-1},v_{m-1})\in W_1(M,T), \tag{2.9}$$

we associate (1.1) with the following problem.

Find $(u_m,v_m)\in W_1(M,T)$ $(m\ge 1)$ which satisfies the linear variational problem

$$\begin{cases}\langle u_m''(t),\phi\rangle+\lambda\langle u_{mx}'(t),\phi_x\rangle+\bar{\mu}_{1m}(t)\langle u_{mx}(t),\phi_x\rangle+\lambda_1\langle u_m'(t),\phi\rangle\\ \qquad\qquad =\langle F_{1m}(t),\phi\rangle+\displaystyle\int_0^t g_1(t-s)\langle u_{mx}(s),\phi_x\rangle ds,\\ \langle v_m''(t),\tilde{\phi}\rangle+\mu_{2m}(t)\left\langle v_{mx}(t),\tilde{\phi}_x\right\rangle+\lambda_2\langle v_m'(t),\tilde{\phi}\rangle\\ \qquad\qquad =\langle F_{2m}(t),\tilde{\phi}\rangle+\displaystyle\int_0^t g_2(t-s)\langle v_{mx}(s),\tilde{\phi}_x\rangle ds,\\ (u_m(0),u_m'(0))=(\tilde{u}_0,\tilde{u}_1),\ \ (v_m(0),v_m'(0))=(\tilde{v}_0,\tilde{v}_1),\end{cases} \tag{2.10}$$

for all $(\phi,\tilde{\phi})\in H_0^1\times V$, a.e. $t\in(0,T)$, where

$$\begin{aligned}\bar{\mu}_{1m}(t)&=\mu_*+\mu_1[u_{m-1}](t)=\mu_*+\mu_1\left(\langle\nabla u_{m-1}(t),\nabla u_{m-1}'(t)\rangle\right),\\ \mu_{2m}(t)&=\mu_2[v_{m-1}](t)=\mu_2\left(\|\nabla v_{m-1}(t)\|^2\right),\\ F_{im}(x,t)&=f_i[u_{m-1},v_{m-1}](x,t)+F_i(x,t)\\ &=f_i(u_{m-1}(x,t),v_{m-1}(x,t))+F_i(x,t),\ i=1,2.\end{aligned} \tag{2.11}$$

Then we have the following theorem that confirms the existence and uniqueness of solutions.

Theorem 2.2. *Let* $(\mathbf{H}_1)-(\mathbf{H}_5)$ *hold. Then there exist constants* M, $T>0$ *such that:*

(i) For $(u_0,v_0)\equiv(0,0)$, *there exists a recurrent sequence* $\{(u_m,v_m)\}\subset W_1(M,T)$ *defined by (2.9)-(2.11).*

(ii) The recurrent sequence $\{(u_m,v_m)\}$ *converges strongly to a pair functions* (u,v) *in the space* $W_1(T)$ *and* $(u,v)\in W_1(M,T)$ *is the unique weak solution of problem (1.1). Moreover, we have the following estimate*

$$\|(u_m,v_m)-(u,v)\|_{W_1(T)}\le C_T K_T^m,\ \textit{for all}\ m\in\mathbb{N}, \tag{2.12}$$

where $K_T\in[0,1)$ *and* C_T *is a contant independent of* m.

Proof. Using the Faedo-Galerkin approximation combined with the Banach fixed point principle, and then performing appropriately priori evaluations and weak compactness arguments, we prove the existence of a linear approximate sequence $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset W_1(M,T)$. Next, we prove that $\{(u_m, v_m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $W_1(T)$, which converges to (u, v) to be the weak solution of (1.1), and also satisfies the estimation of convergent rate as in (2.12). Finally, the uniqueness of the solution is proved by Gronwall's lemma. The detailed proof is the same one as in [20], [23]. Therefore, we omit the details here. □

2.2 General decay of the solution

In what follows, we prove that if

$\mu_* \|\tilde{u}_{0x}\|^2 + \int_0^{\|\tilde{v}_{0x}\|^2} \mu_2(z)\,dz - p\int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x))\,dx > 0$, with $p > 2$, and if the intial energy $E(0)$ and $\|F_1(t)\|^2 + \|F_2(t)\|^2$ are small enough, then every global weak solution of the problem (1.1) is generally decay as $t \to +\infty$. For this purpose, we strengthen the following assumptions

($\bar{\mathrm{H}}_1$) $(\tilde{u}_0, \tilde{v}_0) \in (H_0^1 \cap H^2) \times (V \cap H^2), (\tilde{u}_1, \tilde{v}_1) \in (H_0^1 \cap H^2) \times V$, and $\tilde{v}_{0x} = 0$;

($\bar{\mathrm{H}}_2$) $\mu_1 \in C^1(\mathbb{R})$, and there exists the positive constant $\mu_{1*} < \mu_*$ such that
(i) $\mu_1(y) \geq -\mu_{1*}, \forall y \in \mathbb{R}$, (ii) $y\mu_1(y) \geq 0,\ \forall y \in \mathbb{R}$;

($\bar{\mathrm{H}}_3$) $\mu_2 \in C^1(\mathbb{R}_+)$, and there exist the positive constants $\mu_{2*},\ \chi_*$ such that
(i) $\mu_2(y) \geq \mu_{2*}, \forall y \geq 0$, (ii) $y\mu_2(y) \geq \chi_* \int_0^y \mu_2(z)dz,\ \forall y \geq 0$;

($\bar{\mathrm{H}}_4$) $g_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that
(i) $L_* = \min\left\{\mu_* - \int_0^\infty g_1(s)\,ds,\ \mu_{2*} - \int_0^\infty g_2(s)\,ds\right\} > 0$,
(ii) there exists the function $\xi \in C^1(\mathbb{R}_+)$ such that
$\xi'(t) \leq 0 < \xi(t),\ g_i'(t) \leq -\xi(t)g_i(t) < 0,\ \forall t \geq 0,\ \int_0^\infty \xi(s)\,ds = \infty$;

($\bar{\mathrm{H}}_5$) There exist the function $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ and the constants $\alpha, \beta > 2$, $d_1, \bar{d}_1 > 0$such that
(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = f_1(u,v),\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = f_2(u,v),\ \forall (u,v) \in \mathbb{R}^2$,
(ii) $uf_1(u,v) + vf_2(u,v) \leq d_1\mathcal{F}(u,v),\ \forall (u,v) \in \mathbb{R}^2$,
(iii) $\mathcal{F}(u,v) \leq \bar{d}_1\left(|u|^\alpha + |v|^\beta\right),\ \forall (u,v) \in \mathbb{R}^2$;

($\bar{\mathrm{H}}_6$) $F_1,\ F_2 \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$ such that there exist two constants $C_0,\ \gamma_0 > 0$ satisfying $\|F_1(t)\|^2 + \|F_2(t)\|^2 \leq C_0\exp(-\gamma_0 t),\ \forall t \geq 0$;

($\bar{\mathrm{H}}_7$) $p > \max\left\{2, d_1, \frac{d_1}{\chi_*}\right\}$ and $\frac{\mu_{1*}}{\mu_*} \leq 1 - \frac{d_1}{p}$.

From Theorem 2.2, the problem (1.1) has a local weak solution (u, v) such that

$$\begin{aligned} &(u,v) \in C([0,T]; (H_0^1 \cap H^2) \times V) \cap C^1([0,T]; H_0^1 \times L^2) \\ &\qquad \cap L^\infty(0,T; (H_0^1 \cap H^2) \times (V \cap H^2)), \\ &(u',v') \in C([0,T]; H_0^1 \times L^2) \cap L^\infty(0,T; (H_0^1 \cap H^2) \times V), \\ &(u'',v'') \in L^\infty(0,T; H^2 \cap H_0^1) \times L^\infty(0,T; L^2). \end{aligned} \tag{2.13}$$

Consider the Lyapunov functional as follows

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{2.14}$$

where $\delta > 0$ is chosen later and

$$E(t) = \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g_1 * u)(t) + (g_2 * v)(t) + \tilde{E}\,(t)\right] + \frac{1}{p}I(t),$$

$$\tilde{E}\,(t) = (\mu_* - \bar{g}_1\,(t))\,\|u_x(t)\|^2 - \bar{g}_2\,(t)\,\|v_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)dz,$$

$$I(t) = (g_1 * u)(t) + (g_2 * v)(t) + \tilde{E}\,(t) - p\int_0^1 \mathcal{F}\,(u\,(x,t)\,, v\,(x,t))\,dx,$$

$$\bar{g}_i\,(t) = \int_0^t g_i\,(s)\,ds,\;\; (g_i * \bar{u})(t) = \int_0^t g_i\,(t-s)\,\|\bar{u}_x\,(s) - \bar{u}_x\,(t)\|\,ds,\;\; i = 1, 2,$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \langle v'(t), v(t)\rangle + \frac{\lambda}{2}\|u_x(t)\|^2 + \frac{\lambda_1}{2}\|u(t)\|^2 + \frac{\lambda_2}{2}\|v(t)\|^2.$$

Then, we have the following estimate for $E'(t)$.

Lemma 2.3. *Let (u, v) be a weak solution of (1.1). Then, the energy functional $E(t)$ satisfies*

$$\begin{aligned}
&\textit{(i)}\; E'\,(t) \leq \frac{1}{2}\left(\|F_1\,(t)\| + \|F_2\,(t)\|\right) + \frac{1}{2}\left(\|F_1\,(t)\| + \|F_2\,(t)\|\right)\left(\|u'\,(t)\|^2 + \|v'\,(t)\|^2\right), \\
&\textit{(ii)}\; E'\,(t) \leq -\lambda\,\|u'_x\,(t)\|^2 - \left(\lambda_* - \frac{\varepsilon_1}{2}\right)\left(\|u'\,(t)\|^2 + \|v'\,(t)\|^2\right) \\
&\qquad\qquad -\frac{1}{2}\xi(t)\left[(g_1 * u)\,(t) + (g_2 * v)\,(t)\right] + \frac{1}{2\varepsilon_1}\rho(t),
\end{aligned} \tag{2.15}$$

for all $\varepsilon_1 > 0$, where $\lambda_* = \min\{\lambda_1, \lambda_2\}\,,\; \rho(t) = \|F_1\,(t)\|^2 + \|F_2\,(t)\|^2\,.$

Proof. Multiplying $(1.1)_1$ by $u'(x,t)$, $(1.1)_2$ by $v'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned}
E'\,(t) &= -\lambda_1\,\|u'\,(t)\|^2 - \lambda_2\,\|v'\,(t)\|^2 - \lambda\,\|u'_x\,(t)\|^2 \\
&\quad + \frac{1}{2}\left[(g'_1 * u)\,(t) + (g'_2 * v)\,(t)\right] - \frac{1}{2}\left(g_1\,(t)\,\|u_x(t)\|^2 + g_2\,(t)\,\|v_x(t)\|^2\right) \\
&\quad - \langle u_x(t), u'_x(t)\rangle\mu_1\left(\langle u_x(t), u'_x(t)\rangle\right) + \langle F_1(t), u'(t)\rangle + \langle F_2(t), v'(t)\rangle.
\end{aligned} \tag{2.16}$$

On the other hand

$$\begin{aligned}
\langle F_1(t), u'(t)\rangle &\leq \frac{1}{2\varepsilon_1}\,\|F_1\,(t)\|^2 + \frac{\varepsilon_1}{2}\,\|u'\,(t)\|^2\,, \\
\langle F_2(t), v'(t)\rangle &\leq \frac{1}{2\varepsilon_1}\,\|F_2\,(t)\|^2 + \frac{\varepsilon_1}{2}\,\|v'\,(t)\|^2\,,\;\; \forall\varepsilon_1 > 0, \\
0 &\leq \langle u_x(t), u'_x(t)\rangle\mu_1\left(\langle u_x(t), u'_x(t)\rangle\right), \\
\frac{1}{2}\left[(g'_1 * u)\,(t) + (g'_2 * v)\,(t)\right] &\leq -\frac{1}{2}\left[\xi(t)\,(g_1 * u)\,(t) + \xi(t)\,(g_2 * v)\,(t)\right],
\end{aligned} \tag{2.17}$$

then it follows from (2.16), (2.17) that the inequalities (2.15) are valid. Lemma 2.3 is proved. □

Now, putting

$$R_* = \left(\frac{2pE_*}{(p-2)L_*}\right)^{1/2},\;\; E_* = \left(E(0) + \frac{1}{2}\rho\right)\exp\left(\rho\right),\;\; \rho = \int_0^\infty \left(\|F_1\,(t)\| + \|F_2\,(t)\|\right)dt,$$

we obtain the following lemma.

Lemma 2.4. *Assume that $(\bar{H}_1)-(\bar{H}_7)$ hold. Let $(\tilde{u}_0,\tilde{v}_0)\in(H^2\cap H_0^1)\times(H^2\cap V)$ such that $I(0)>0$ and the initial energy $E(0)$ satisfies*

$$\begin{cases} (i)\ \bar{g}_1(\infty)+d_1\bar{d}_1\max\left\{R_*^{\alpha-2},R_*^{\beta-2}\right\}<\mu_*-\mu_{1*}, \\ (ii)\ \left(1-\dfrac{d_1}{p}\right)\bar{g}_1(\infty)+\dfrac{d_1}{p}\bar{g}_2(\infty)+d_1\bar{d}_1\max\left\{R_*^{\alpha-2},R_*^{\beta-2}\right\} \\ \qquad <\left(1-\dfrac{d_1}{p}\right)\mu_*+\dfrac{d_1\mu_{2*}}{p}-\mu_{1*}, \\ (iii)\ \dfrac{d_1}{p}\bar{g}_1(\infty)+\left(1-\dfrac{d_1}{p}\right)\bar{g}_2(\infty)+d_1\bar{d}_1\max\left\{R_*^{\alpha-2},R_*^{\beta-2}\right\} \\ \qquad <\dfrac{d_1\mu_*}{p}+\left(1-\dfrac{d_1}{p\chi_*}\right)\mu_{2*}, \\ (iv)\ \bar{g}_2(\infty)+d_1\bar{d}_1\max\left\{R_*^{\alpha-2},R_*^{\beta-2}\right\}<\left(\dfrac{d_1}{p}+1-\dfrac{d_1}{p\chi_*}\right)\mu_{2*}, \\ (v)\ \eta^*=L_*-p\bar{d}_1\max\left(R_*^{\alpha-2},R_*^{\beta-2}\right)>0. \end{cases} \tag{2.18}$$

Then $I(t)>0$, for all $t\geq 0$.

Proof. By the continuity of $I(t)$ and $I(0)>0$, there exists $\tilde{T}>0$ such that

$$I(t)=I(u(t),v(t))>0,\ \forall t\in[0,\tilde{T}]. \tag{2.19}$$

It is easy to see that

$$\tilde{E}(t)\geq L_*\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right). \tag{2.20}$$

From (2.19) and (2.20), this implies that

$$E(t)\geq\frac{1}{2}\left(\|u'(t)\|^2+\|v'(t)\|^2\right)+\frac{(p-2)L_*}{2p}\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right),\ \forall t\in[0,\tilde{T}]. \tag{2.21}$$

Using Lemma 2.3, (2.21), and Gronwall's inequality, we obtain

$$\begin{aligned} &\|u_x(t)\|^2+\|v_x(t)\|^2\leq\frac{2pE(t)}{(p-2)L_*}\leq\frac{2pE_*}{(p-2)L_*}\equiv R_*^2,\ \forall t\in[0,\tilde{T}], \\ &\|u'(t)\|^2+\|v'(t)\|^2\leq 2E(t)\leq 2E_*,\ \forall t\in[0,\tilde{T}]. \end{aligned} \tag{2.22}$$

Then, from the assumption ($\bar{\text{H}}_5$,(iii)) and (2.22), the result is

$$\begin{aligned} p\int_0^1\mathcal{F}(u(x,t),v(x,t))\,dx &\leq p\bar{d}_1\left(\|u(t)\|_{L^\alpha}^\alpha+\|v(t)\|_{L^\beta}^\beta\right) \\ &\leq p\bar{d}_1\left(\|u_x(t)\|^\alpha+\|v_x(t)\|^\beta\right) \\ &\leq p\bar{d}_1\left(R_*^{\alpha-2}\|u_x(t)\|^2+R_*^{\beta-2}\|v_x(t)\|^2\right) \\ &\leq p\bar{d}_1\max\left\{R_*^{\alpha-2},R_*^{\beta-2}\right\}\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right). \end{aligned} \tag{2.23}$$

Therefore, we get

$$I(t)\geq\eta^*\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right)+(g_1*u)(t)+(g_2*v)(t)\geq 0,\ \forall t\in[0,\tilde{T}], \tag{2.24}$$

where the constant $\eta^*>0$ is defined as in (2.18).

Next, we put $T_\infty=\sup\{T>0: I(t)>0,\ \forall t\in[0,T]\}$. Suppose that $T_\infty<+\infty$ then, because of the continuity of $I(t)$, we have $I(T_\infty)\geq 0$. In case of $I(T_\infty)>0$, by the same arguments as above, we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, \tilde{T}_\infty]$. We obtain a contradiction to the definition of T_∞. In case of $I(T_\infty) = 0$, it implies from (2.24) that

$$0 = I(T_\infty) \geq \eta^* \left(\|u_x(T_\infty)\|^2 + \|v_x(T_\infty)\|^2\right) + (g_1 * u)(T_\infty) + (g_2 * v)(T_\infty) \geq 0.$$

Therefore

$$\|u(T_\infty)\| = \|v(T_\infty)\| = (g_1 * u)(T_\infty) = (g_2 * v)(T_\infty) = 0.$$

By the fact that the function $s \longmapsto g_1(T_\infty - s)\|u_x(T_\infty) - u_x(s)\|^2$ is continuous on $[0, T_\infty]$ and $g_1(T_\infty - s) > 0$, $\forall s \in [0, T_\infty]$, we have

$$(g_1 * u)(T_\infty) = \int_0^{T_\infty} g_1(T_\infty - s)\|u_x(s)\|^2 ds = 0,$$

it follows that $\|u_x(s)\|^2 = 0$, $\forall s \in [0, T_\infty]$, it means that $u(0) = 0$. Similary, $v(0) = 0$. It leads to $I(0) = 0$. We get a contradiction with $I(0) > 0$. Consequently, $T_\infty = +\infty$, i.e., $I(t) > 0$, $\forall t \geq 0$. Lemma 2.4 is proved completely. □

Next, we put

$$E_1(t) = \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + (g_1 * u)(t) + (g_2 * v)(t) + I(t). \tag{2.25}$$

Then, we have the following lemma.

Lemma 2.5. *There exist the positive constants β_1, $\bar{\beta}_1$, β_2, $\bar{\beta}_2$ such that*

$$\begin{aligned} &(i) \quad \beta_1 E_1(t) \leq \mathcal{L}(t) \leq \beta_2 E_1(t), \ \forall t \geq 0, \\ &(ii) \quad \bar{\beta}_1 E_1(t) \leq E(t) \leq \bar{\beta}_2 E_1(t), \ \forall t \geq 0. \end{aligned} \tag{2.26}$$

Proof. The functional $\mathcal{L}(t)$ can be written as follows

$$\begin{aligned} \mathcal{L}(t) = &\frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{p-2}{2p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{p-2}{2p}\tilde{E}(t) + \frac{1}{p}I(t) \\ &+\delta\left(\langle u'(t), u(t)\rangle + \langle v'(t), v(t)\rangle\right) + \frac{\delta}{2}\left(\lambda\|u_x(t)\|^2 + \lambda_1\|u(t)\|^2 + \lambda_2\|v(t)\|^2\right). \end{aligned} \tag{2.27}$$

From the following inequalities

$$\begin{aligned} |\langle u'(t), u(t)\rangle| &\leq \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2, \\ |\langle v'(t), v(t)\rangle| &\leq \frac{1}{2}\|v'(t)\|^2 + \frac{1}{2}\|v_x(t)\|^2, \end{aligned}$$

we get

$$\begin{aligned} \mathcal{L}(t) \geq &\frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{p-2}{2p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{p}I(t) \\ &+ \frac{(p-2)L_*}{2p}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) - \delta\left(\frac{\|u'(t)\|^2 + \|u_x(t)\|^2}{2} + \frac{\|v'(t)\|^2 + \|v_x(t)\|^2}{2}\right) \\ = &\frac{1-\delta}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \left(\frac{(p-2)L_*}{2p} - \frac{\delta}{2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ &+ \frac{p-2}{2p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{p}I(t) \geq \beta_1 E_1(t), \end{aligned}$$

where $\beta_1 = \min\left\{\frac{1-\delta}{2},\ \frac{p-2}{2p},\ \frac{(p-2)L_*}{2p} - \frac{\delta}{2},\ \frac{1}{p}\right\}$ and $0 < \delta < \min\left\{1;\ \frac{(p-2)L_*}{p}\right\}$.

Similarly, we can prove that

$$\begin{aligned}\mathcal{L}(t) \leq &\frac{1+\delta}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{p-2}{2p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] \\ &+ \frac{p-2}{2p}\left(\mu_* \|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)dz\right) + \delta\frac{\|u_x(t)\|^2 + \|v_x(t)\|^2}{2} \\ &+ \frac{\delta}{2}\left[(\lambda + \lambda_1)\|u_x(t)\|^2 + \lambda_2\|v_x(t)\|^2\right] + \frac{1}{p}I(t).\end{aligned}$$

Put $\mu^*_{2\max} = \max\limits_{0\leq z\leq R_*^2} \mu_2(z)$, we have $\int_0^{\|v_x(t)\|^2} \mu_2(z)dz \leq \mu^*_{2\max}\|v_x(t)\|^2$, hence

$$\begin{aligned}\mathcal{L}(t) \leq &\frac{1+\delta}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \left[\frac{(p-2)}{2p}\mu_* + \frac{\delta}{2}(1+\lambda+\lambda_1)\right]\|u_x(t)\|^2 \\ &+ \left[\frac{p-2}{2p}\mu^*_{2\max} + \frac{\delta}{2}(1+\lambda_2)\right]\|v_x(t)\|^2 + \frac{p-2}{2p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{p}I(t) \\ &\leq \beta_2 E_1(t),\end{aligned}$$

where $\beta_2 = \max\left\{\frac{1+\delta}{2},\ \frac{(p-2)\mu_*}{2p} + \frac{\delta(1+\lambda+\lambda_1)}{2},\ \frac{(p-2)\mu^*_{2\max}}{2p} + \frac{\delta(1+\lambda_2)}{2}\right\}$. Part (i) of Lemma 2.5 is done. It is similar to prove Part (ii). □

Lemma 2.6. *The functional $\psi(t)$ satisfies the following estimation*

$$\begin{aligned}\psi'(t) \leq &\|u'(t)\|^2 + \|v'(t)\|^2 + \left(\frac{d_2}{p} + \frac{1}{2\varepsilon_3}\right)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] \qquad (2.28)\\ &- \frac{\delta_1 d_2}{p}I(t) + \frac{1}{2\varepsilon_2}\rho(t) \\ &- \left[\frac{(1-\delta_1)d_2\eta^*}{p} - \frac{\varepsilon_2}{2} + \left(1 - \frac{d_2}{p}\right)\mu_* - \mu_{1*} - \left(1 - \frac{d_2}{p} + \frac{\varepsilon_3}{2}\right)\bar{g}_1(\infty)\right]\|u_x(t)\|^2 \\ &- \left[\frac{(1-\delta_1)d_2\eta^*}{p} - \frac{\varepsilon_2}{2} - \left(1 - \frac{d_2}{p} + \frac{\varepsilon_3}{2}\right)\bar{g}_2(\infty) + \left(1 - \frac{d_2}{p\chi_*}\right)\mu_{2*}\right]\|v_x(t)\|^2, \qquad (2.29)\end{aligned}$$

for all $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\delta_1 \in (0,1)$, and $\rho(t) = \|F_1(t)\|^2 + \|F_2(t)\|^2$.

Proof. By multiplying $(1.1)_1$ by $u(x,t)$, $(1.1)_2$ by $v(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) = &\|u'(t)\|^2 + \|v'(t)\|^2 - \left[\mu_* + \mu_1(\langle u_x(t), u_x'(t)\rangle)\ \right]\|u_x(t)\|^2 \qquad (2.30)\\ &- \|v_x(t)\|^2\mu_2\left(\|v_x(t)\|^2\right) + \langle f_1(u(t), v(t)), u(t)\rangle \\ &+ \langle f_2(u(t), v(t)), v(t)\rangle + \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle \\ &+ \int_0^t g_1(t-s)\langle u_x(s), u_x(t)\rangle ds + \int_0^t g_2(t-s)\langle v_x(s), v_x(t)\rangle ds.\end{aligned}$$

Note that

$$
\begin{aligned}
&-\mu_1\left(\langle u_x(t), u_x'(t)\rangle\right) \le \mu_{1*}, \\
&\langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle \le \frac{\varepsilon_2}{2}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) + \frac{1}{2\varepsilon_2}\rho(t), \\
&\int_0^{\|v_x(t)\|^2} \mu_2(z)dz \le \frac{1}{\chi_*}\|v_x(t)\|^2 \mu_2\left(\|v_x(t)\|^2\right), \\
&I(t) \ge \eta^*\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right), \\
&\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \le d_1 \int_0^1 \mathcal{F}\left(u(x,t), v(x,t)\right)dx \\
&\qquad = \frac{d_1}{p}\tilde{E}(t) + \frac{d_1}{p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] - \frac{d_1}{p}I(t) \\
&\le \frac{d_1}{p}\tilde{E}(t) + \frac{d_1}{p}\left[(g_1 * u)(t) + (g_2 * v)(t)\right] - \frac{\delta_1 d_1}{p}I(t) - \frac{(1-\delta_1)d_1\eta^*}{p}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right).
\end{aligned}
\tag{2.31}
$$

It's not difficult, we have

$$
\begin{aligned}
&\int_0^t g_1(t-s)\langle u_x(s), u_x(t)\rangle ds \le \left(1 + \frac{\varepsilon_3}{2}\right)\bar{g}_1(t)\|u_x(t)\|^2 + \frac{1}{2\varepsilon_3}(g_1 * u)(t), \\
&\int_0^t g_2(t-s)\langle v_x(s), v_x(t)\rangle ds \le \left(1 + \frac{\varepsilon_3}{2}\right)\bar{g}_2(t)\|v_x(t)\|^2 + \frac{1}{2\varepsilon_3}(g_2 * v)(t), \forall \varepsilon_3 > 0.
\end{aligned}
\tag{2.32}
$$

Then, it follows from (2.30)-(2.32) that the inequality (2.29) is valid. Lemma 2.6 is proved. □

Based on the above results, we can prove the main result in this section as follows.

Theorem 2.7. *Assume that $(\bar{H}_1) - (\bar{H}_7)$ hold. Let $(\tilde{u}_0, \tilde{v}_0) \in (H^2 \cap H_0^1) \times (H^2 \cap V)$ such that $I(0) > 0$ and the initial energy $E(0)$ satisfy (2.18). Then, any global weak solution of problem (1.1) is generally decay, i.e., there exist positive constants $\bar{C}$, $\bar{\gamma}$ such that*

$$
\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \le \bar{C}\exp\left(-\bar{\gamma}\int_0^t \xi(s)ds\right), \ \forall t \ge 0. \tag{2.33}
$$

Proof. It follows from (2.14), $(2.15)_{ii}$ and (2.29) that

$$
\begin{aligned}
\mathcal{L}'(t) \le &-\left(\lambda_* - \frac{\varepsilon_1}{2} - \delta\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - \frac{1}{2}\xi(t)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] \\
&+ \delta\left(\frac{d_1}{p} + \frac{1}{2\varepsilon_3}\right)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] - \delta\theta_1\|u_x(t)\|^2 - \delta\theta_2\|v_x(t)\|^2 \\
&- \frac{\delta\delta_1 d_1}{p}I(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t),
\end{aligned}
\tag{2.34}
$$

where

$$
\begin{aligned}
\theta_1 = \theta_1(\delta_1, \varepsilon_2, \varepsilon_3) = &\frac{(1-\delta_1)d_1\eta^*}{p} - \frac{\varepsilon_2}{2} + \left(1 - \frac{d_1}{p}\right)\mu_* - \mu_{1*} \\
&- \left(1 - \frac{d_1}{p} + \frac{\varepsilon_3}{2}\right)\bar{g}_1(\infty), \\
\theta_2 = \theta_2(\delta_1, \varepsilon_2, \varepsilon_3) = &\frac{(1-\delta_1)d_1\eta^*}{p} - \frac{\varepsilon_2}{2} - \left(1 - \frac{d_1}{p} + \frac{\varepsilon_3}{2}\right)\bar{g}_2(\infty) \\
&+ \left(1 - \frac{d_1}{p\chi_*}\right)\mu_{2*}.
\end{aligned}
$$

We have

$$\begin{aligned}
\lim_{\delta_1\to 0_+,\varepsilon_2\to 0_+,\varepsilon_3\to 0_+} \theta_1(\delta_1,\varepsilon_2,\varepsilon_3) &= \frac{d_1\eta^*}{p} + \left(1-\frac{d_1}{p}\right)\mu_* - \mu_{1*} - \left(1-\frac{d_1}{p}\right)\bar{g}_1(\infty) \\
&= \frac{d_1}{p}\left[L_* - \tilde{R}_* + \left(\frac{p}{d_1}-1\right)(\mu_* - \bar{g}_1(\infty)) - \frac{p\mu_{1*}}{d_1}\right] \\
&\equiv \theta_1^*, \\
\lim_{\delta_1\to 0_+,\varepsilon_2\to 0_+,\varepsilon_3\to 0_+} \theta_2(\delta_1,\varepsilon_2,\varepsilon_3) &= \frac{d_1\eta^*}{p} - \left(1-\frac{d_1}{p}\right)\bar{g}_2(\infty) + \left(1-\frac{d_1}{p\chi_*}\right)\mu_{2*} \\
&= \frac{d_1}{p}\left[L_* - \tilde{R}_* - \left(\frac{p}{d_1}-1\right)\bar{g}_2(\infty) + \left(\frac{p}{d_1}-\frac{1}{\chi_*}\right)\mu_{2*}\right] \\
&\equiv \theta_2^*,
\end{aligned}$$

where $\eta^* = L_* - \tilde{R}_*$, $\tilde{R}_* = p\bar{d}_1 \max\left\{R_*^{\alpha-2}, R_*^{\beta-2}\right\}$.

Note that, we have $\theta_1^* > 0$ provided by the conditions (2.18) (i), (ii), and also have $\theta_2^* > 0$ by (2.18) (iii), (iv). Thus, we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ small enough such that

$$\theta_1 = \theta_1(\delta_1,\varepsilon_2,\varepsilon_3) > 0, \quad \theta_2 = \theta_2(\delta_1,\varepsilon_2,\varepsilon_3) > 0. \tag{2.35}$$

Moreover, we continue by choosing $\varepsilon_1 > 0$, $\delta > 0$ small enough such that

$$\bar{\theta}_1 = \lambda_* - \frac{\varepsilon_1}{2} - \delta > 0, \ 0 < \delta < \min\left\{1; \ \frac{(p-2)L_*}{p}\right\}. \tag{2.36}$$

Put

$$\bar{\theta}_2 = \min\{\delta\theta_1, \delta\theta_2\}, \ \bar{\theta}_3 = \frac{\delta\delta_1 d_1}{p}, \ \bar{\theta}_4 = \delta\left(\frac{d_1}{p} + \frac{1}{2\varepsilon_3}\right), \ \bar{\theta}_* = \min\left\{\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3\right\}. \tag{2.37}$$

From (2.34), this implies that

$$\begin{aligned}
\mathcal{L}'(t) \le& -\bar{\theta}_1\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - \bar{\theta}_2\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\
&- \bar{\theta}_3 I(t) + \bar{\theta}_4\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t) \\
\le& -\bar{\theta}_*\left[\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + I(t)\right] \\
&+ \bar{\theta}_4\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t) \\
\le& -\bar{\theta}_* E_1(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t) \\
\le& -\frac{\bar{\theta}_*}{\bar{\beta}_2} E(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho(t).
\end{aligned} \tag{2.38}$$

Combining $(2.15)_{ii}$ and (2.38), we conclude that

$$\begin{aligned}
\xi(t)\mathcal{L}'(t) &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)\xi(t)\left[(g_1 * u)(t) + (g_2 * v)(t)\right] \\
&\quad + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\rho(t) \\
&\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)\left(-E'(t) + \frac{1}{2\varepsilon_1}\rho(t)\right) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\rho(t) \\
&= -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) - 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) + \frac{1}{2}\left[\frac{\bar{\theta}_* + \bar{\theta}_4}{\varepsilon_1} + \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\right]\rho(t) \\
&\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) - 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) + \bar{C}_0 e^{-\gamma_0 t},
\end{aligned} \tag{2.39--2.40}$$

where $\bar{C}_0 = \frac{1}{2}\left[\frac{\bar{\theta}_*+\bar{\theta}_4}{\varepsilon_1} + \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\right]C_0$.

For convenience, we continue to define the new functional

$$L(t) = \xi(t)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t). \tag{2.41}$$

It is easy to see that

$$\begin{aligned}
L(t) &\leq \xi(0)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \\
&\leq \xi(0)\beta_2 E_1(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \\
&\leq \left[\frac{\beta_2}{\bar{\beta}_1}\xi(0) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)\right]E(t) \equiv \hat{\beta}_2 E(t).
\end{aligned} \tag{2.42}$$

By direct computations, it yields

$$\begin{aligned}
L'(t) &= \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) \\
&\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) + \bar{C}_0 e^{-\gamma_0 t} \leq -\frac{\bar{\theta}_*}{\bar{\beta}_2\hat{\beta}_2}\xi(t)L(t) + \bar{C}_0 e^{-\gamma_0 t}.
\end{aligned} \tag{2.43}$$

Choosing $0 < \bar{\gamma} < \min\left\{\frac{\bar{\theta}_*}{\bar{\beta}_2\hat{\beta}_2}, \frac{\gamma_0}{\xi(0)}\right\}$, from (2.43), we get

$$L'(t) + \bar{\gamma}\xi(t)L(t) \leq \bar{C}_0 e^{-\gamma_0 t}. \tag{2.44}$$

Integrating (2.44) with respect to time variable, we obtain

$$L(t) \leq \left(L(0) + \frac{\bar{C}_0}{\gamma_0 - \bar{\gamma}\xi(0)}\right)\exp\left(-\bar{\gamma}\int_0^t \xi(s)ds\right). \tag{2.45}$$

On the other hand, we have

$$\begin{aligned}
L(t) &= \xi(t)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \\
&\geq 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \geq 2\left(\bar{\theta}_* + \bar{\theta}_4\right)\bar{\beta}_1 E_1(t), \\
E_1(t) &\geq \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2.
\end{aligned} \tag{2.46}$$

Combining (2.45) and (2.46), we get (2.33). The proof of Theorem 2.7 is completed. □

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