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Existence and general decay estimates of the Dirichlet problem for a nonlinear Kirchhoff wave equation in an annular with viscoelastic term

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ABSTRACT
This paper is devoted to the study of a Kirchhoff wave equation with a viscoelastic term in an annular associated with homogeneous Dirichlet conditions. At first, by applying the Faedo-Galerkin, we prove existence and uniqueness of the solution of the problem considered. Next, by constructing Lyapunov functional, we establish a sufficient condition such that any global weak solution is general decay as $t \to +\infty$.
Keywords: *Faedo-Galerkin method, Nonlinear Kirchhoff wave equation, local existence, general decay.*

1 Introduction

In this paper, we study the following Dirichlet problem for a Kirchhoff wave equation with a viscoelastic term in an annular

$$\begin{cases} u_{tt} - \mu\left(t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right) \dfrac{1}{x}\dfrac{\partial}{\partial x}(xu_x) + \displaystyle\int_0^t g(t-s)\dfrac{1}{x}\dfrac{\partial}{\partial x}(xu_x(s))\,ds \\ \qquad\qquad = f(x,t,u,u_x,u_t), \ \rho < x < 1, \ 0 < t < T, \\ u(\rho,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $\rho \in (0,1)$ is a given constant and μ, g, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions to be specified later. In Eq. (1.1), the nonlinear term $\mu\left(t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right)$ depends on the integrals

$$\|u_x(t)\|_0^2 = \int_0^1 xu_x^2(x,t)\,dx, \ \ \|u(t)\|_0^2 = \int_0^1 xu^2(x,t)\,dx. \tag{1.2}$$

Eq.(1.1) herein is the bidimensional nonlinear wave equation describing the nonlinear vibrations of the annular membrane $\Omega = \{(x,y) : \rho^2 < x^2 + y^2 < 1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time.

The conditions $u(\rho,t) = u(1,t) = 0$, on the boundaries $\Gamma_\rho = \{(x,y) : x^2+y^2 = \rho^2\}$ and $\Gamma_1 = \{(x,y) : x^2+y^2 = 1\}$ of the annular membrane is fixed.

It is known that Kirchhoff [6] first investigated the following nonlinear vibration of an elastic string

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L} \int_0^L \left| \frac{\partial u}{\partial y}(y,t) \right|^2 dy \right) u_{xx}, \tag{1.3}$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension.

In [3], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0} \int_0^L u^2(y,t)dy \right) u_{xx} = 0, \tag{1.4}$$

where $u(x,t)$ is the $x-$derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross - section of a string, L is the length of a string and and ρ is the density of a material. Clearly, if properties of a material depends on x and t, there is a hyperbolic equation of the type (Larkin [7])

$$u_{tt} - B\left(x, t, \int_0^1 u^2\left(y,t\right) dy \right) u_{xx} = 0. \tag{1.5}$$

The Kirchhoff - Carrier equations of the form Eq.(1.1) received much attention. We refer the reader to, e.g., Cavalcanti et al. [1], [2], Ebihara, Medeiros and Miranda [4], Miranda et al. [15], Lasiecka and Ong [8], Hosoya, Yamada [5], Larkin [7], Long et al. [10] - [12], Medeiros [14], Menzala [16], Messaoudi [17], Ngoc et al. [18]- [22], Park et al. [23], [24], Rabello et al. [25], Santos and et al. [26], Truong et al. [29]. In these works, the results concerning local existence, global existence, asymptotic expansion, asymptotic behavior, decay and blow-up of solutions have been examined.

Recently, Gongwei Liu [13] studied the damped wave equation of Kirchhoff type with initial and Dirichlet boundary condition

$$\begin{cases} u_{tt} - M\left(\|\nabla u\left(t\right)\|^2 \right) \Delta u + u_t = g\left(u\right) \ \text{trong } \Omega \times (0,\infty), \\ u\left(x,0\right) = u_0\left(x\right), \quad u_t\left(x,0\right) = u_1\left(x\right), \ x \in \Omega, \\ u\left(x,t\right) = 0 \ \ \text{trên } \partial\Omega \times (0,\infty), \end{cases} \tag{1.6}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$. Under the assumption that g is a function with exponential growth at infinity, the author proved global existence and the decay property as well as blow-up of solutions in finite time under suitable conditions.

In [15], Miranda and Jutuca dealt with the existence and uniqueness of solutions and exponential decay of solutions of an initial-homogeneous boundary value problem for the Kirchhoff equation.

In [1], [2], Cavalcanti also studied the existence and uniform decay of solutions of the Kirchhoff -Carrier equation.

In [29], Truong et al. concerned with the global existence and regularity of weak solutions to the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f\left(x,t\right), \ 0 < x < 1, \ t > 0, \tag{1.7}$$

with the initial conditions as in $(1.1)_3$ and two-point boundary conditions. The exponential decay of solutions was given there by using Lyapunov method.

Motivated by the above works, we study the unique existence and general decay of the solution for Prob. (1.1) under suitable conditions on f, μ and initial data. Our paper is organized as follows.

In Section 2, we present preliminaries with the notations, definitions, appropriate spaces and required lemmas. In Section 3, we prove the existence and uniqueness of a weak solution for Prob. (1.1). Finally, Section 4 is devoted to considering Prob. (1.1) in the case of $\mu = \mu\left(\|u_x(t)\|_0^2\right)$, $f(x,t,u,u_x,u_t) := -\lambda u_t + f(u) + F(x,t)$, where $\lambda > 0$ is constant, and f, F are given functions. In this section, we verify that, if $\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)dz - p\int_\rho^1 xdx \int_0^{\tilde{u}_0(x)} f(z)dz > 0$, and the initial energy $E(0) = \frac{1}{2}\|\tilde{u}_1\|_0^2 + \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)\,dz - \int_\rho^1 xdx\int_0^{\tilde{u}_0(x)} f(z)dz$ and F are small enough, then any global weak solution of Prob.(1.1) is general decay as $t \to +\infty$. In the proofs, we use the multiplier technique combined with a suitable Lyapunov functional. Our results can be regarded as an extension and improvement of the corresponding results of [7], [10] - [12], [18] - [22], [28], [29].

2 Preliminaries

First, we put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $0 < \rho < 1$, $T > 0$ and denote the usual function spaces used throughout the paper by the notations $C^m = C^m\left(\overline{\Omega}\right)$, $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$, $W^{m,p} = W^{m,p}(\Omega)$.

We remark that L^2, H^1, H^2 are the Hilbert spaces with respect to the corresponding scalar products

$$\langle u,v\rangle = \int_\rho^1 xu(x)\,v(x)\,dx,\ \ \langle u,v\rangle + \langle u_x,v_x\rangle,\ \ \langle u,v\rangle + \langle u_x,v_x\rangle + \langle u_{xx},v_{xx}\rangle. \tag{2.1}$$

The norms in L^2, H^1 and H^2 induced by the corresponding scalar products (2.1) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

We then have the following lemma.

Lemma 2.1. *The embedding $H_0^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and for all $v \in H_0^1$, we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \le \sqrt{1-\rho}\,\|v_x\|$,

(ii) $\|v\| \le \frac{1-\rho}{\sqrt{2}}\|v_x\|$,

(iii) $\|v\|_0 \le \frac{1-\rho}{\sqrt{2\rho}}\|v_x\|_0$.

We denote $\|\cdot\|_X$ for the norm in the Banach space X and call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\int_0^T \|u(t)\|_X^p\,dt < +\infty,\ 1 \le p < \infty,$$

and

$$\exists M > 0 : \|u(t)\|_X \le M \text{ a.e } t \in (0,T),\ p = \infty$$

with

$$||u||_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p \, dt\right)^{1/p}, & 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} ||u(t)||_X, & p = \infty. \end{cases}$$

We also denote

$$u(x,t),\ \frac{\partial u}{\partial t}(x,t),\ \frac{\partial^2 u}{\partial t^2}(x,t),\ \frac{\partial u}{\partial x}(x,t),\ \frac{\partial^2 u}{\partial x^2}(x,t)$$

by

$$u(t),\ u_t(t) = \dot{u}(t),\ u_{tt}(t) = \ddot{u}(t),\ u_x(t) = \nabla u(t),\ u_{xx}(t) = \Delta u(t),$$

respectively

$$u(x,t),\ \frac{\partial u}{\partial t}(x,t),\ \frac{\partial^2 u}{\partial t^2}(x,t),\ \frac{\partial u}{\partial x}(x,t),\ \frac{\partial^2 u}{\partial x^2}(x,t).$$

3 Main results

3.1 The existence and uniqueness of a weak solution

The weak fomulation of Prob. (1.1) can be given in the following manner: Find $u \in \hat{W}_T = \{u \in L^\infty(0,T;H^2 \cap H_0^1) : u' \in L^\infty(0,T;H_0^1) \text{ and } u'' \in L^\infty(0,T;L^2)\}$, such that u satisfies the following variational problem:

$$\langle u''(t), v\rangle + \mu[u](t) \langle u_x(t), v_x\rangle = \int_0^t g\,(t-s) \langle u_x(s), v_x\rangle \, ds + \langle f[u](t), v\rangle\,, \tag{3.1}$$

for all $v \in H_0^1$, and a.e., $t \in (0,T)$, and the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$\begin{aligned} f[u](x,t) &= f\left(x,t,u(x,t),u_x(x,t),u_t(x,t)\right), \\ \mu[u](t) &= \mu\left(t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right). \end{aligned} \tag{3.3}$$

Remark 3.1. *The weak solutions have following properties*

$$\begin{aligned} u &\in L^\infty(0,T;H^2 \cap H_0^1) \cap C^0([0,T];H_0^1) \cap C^1([0,T];L^2), \\ u' &\in L^\infty(0,T;H_0^1) \cap C^0([0,T];L^2), \\ u'' &\in L^\infty(0,T;L^2). \end{aligned}$$

Fix $T^* > 0$ and we make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in (H^2 \cap H_0^1) \times H_0^1$;
(H_2) $\mu \in C^1([0,T^*] \times \mathbb{R}_+^2)$, and there exist a positive constant μ_{1*} such that
$\mu(t,y,z) \geq \mu_{1*}$, $\forall (t,y,z) \in [0,T^*] \times \mathbb{R}_+^2$;
(H_3) $g \in C^1([0,T^*])$;
(H_4) $f \in C^1([\rho,1] \times [0,T^*] \times \mathbb{R}^3)$ such that
$f(\rho,t,0,y,0) = f(1,t,0,y,0) = 0$, $\forall (t,y) \in [0,T^*] \times \mathbb{R}$.

For each $T \in (0, T^*]$, we put

$$W_T = \{v \in L^\infty(0,T;H^2 \cap H_0^1) : v' \in L^\infty(0,T;H_0^1),\, v'' \in L^2(0,T;L^2)\}, \qquad (3.4)$$
$$W_1(T) = C^0([0,T];H_0^1) \cap C^1([0,T];L^2),$$

are Banach spaces with respect to the norms (Lions [9])

$$\|v\|_{W_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)}\,;\ \|v'\|_{L^\infty(0,T;H_0^1)}\,;\ \|v''\|_{L^2(0,T;L^2)}\}, \qquad (3.5)$$
$$\|v\|_{W_1(T)} = \|v\|_{C^0([0,T];H_0^1)} + \|v'\|_{C^0([0,T];L^2)}\,.$$

For each $M > 0$, we denote

$$W(M,T) = \{v \in W_T : \|v\|_{W_T} \le M\}, \qquad (3.6)$$
$$W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}.$$

We establish the linear recurrent sequence $\{u_m\}$ as follows:
We shall choose as first initial term $u_0 \equiv 0$ and suppose that

$$u_{m-1} \in W_1(M,T). \qquad (3.7)$$

We find $u_m \in W_1(M,T)$ is a solution of variational problem associated Problem (1.1) as follow:

$$\begin{cases} \langle u_m''(t), v\rangle + \mu_m(t)\,\langle u_{mx}(t), v_x\rangle = \displaystyle\int_0^t g(t-s)\,\langle u_{mx}(s), v_x\rangle\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \langle F_m(t), v\rangle\,,\ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \qquad (3.8)$$

where

$$\mu_m(t) = \mu[u_{m-1}](t) = \mu(t, \|u_{m-1}(t)\|_0^2\,, \|\nabla u_{m-1}(t)\|_0^2), \qquad (3.9)$$
$$F_m(x,t) = f[u_{m-1}](x,t) = f\left(x, t, u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t)\right).$$

Then we have the following theorem concerning the existence and uniqueness of a weak solution.

Theorem 3.2. *Let $(H_1) - (H_4)$ hold. Then there exist constants $M > 0$, $T > 0$ such that:*

(i) *Problem (1.1) has a unique weak solution $u \in W_1(M,T)$.*

(ii) *The linear recurrent sequence $\{u_m\}$ defined by (3.7)-(3.9) converges to the solution u strongly in the space $W_1(T)$ with the estimate*

$$\|u_m - u\|_{W_1(T)} \le \frac{M}{1-k_T} k_T^m, \text{ for all } m \in \mathbb{N}, \qquad (3.10)$$

where $k_T \in [0,1)$ is a constant independent of m.

Proof. The proof is similar to the argument in [28], so we omit the details. □

3.2 General decay of the solution

In this section, Prob. (1.1) is considered with $\mu := \mu\left(\|u_x(t)\|_0^2\right)$, $f(x,t,u,u_x,u_t) := -\lambda u_t + f(u) + F(x,t)$, so Prob. (1.1) becomes the following

$$\begin{cases} u_{tt} - \mu\left(\|u_x(t)\|_0^2\right)\left(u_{xx} + \frac{1}{x}u_x\right) + \int_0^t g(t-s)\left(u_{xx}(x,s) + \frac{1}{x}u_x(x,s)\right)ds \\ \qquad + \lambda u_t = f(u) + F(x,t), \;\; \rho < x < 1, t > 0, \\ u(\rho,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \;\; u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{3.11}$$

where $0 < \rho < 1$, $\lambda > 0$ are the given constants, F, μ, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions. Here, we shall prove that any global weak solution $u(t)$ of Prob. (3.11) is general decay as $t \to +\infty$, i.e., there exist positive constants $\bar{C}$, γ and a positive function ξ such that

$$\|u'(t)\|_0^2 + \|u_x(t)\|_0^2 \le \bar{C}\exp\left(-\overline{\gamma}\int_0^t \xi(s)ds\right), \text{ for all } t \ge 0, \tag{3.12}$$

and $\int_0^{+\infty} \xi(s)ds = +\infty.$

First, we suppose that

- (H_1) $\tilde{u}_0 \in H^2 \cap H_0^1$, $\tilde{u}_1 \in H_0^1$;
- $(\hat{H}_2)$ $\mu \in C^1(\mathbb{R}_+)$ and there exists a positive constant μ_* such that $\mu(z) \ge \mu_*$, $\forall z \ge 0$;
- $(\hat{H}_3)$ $g \in C^1(\mathbb{R}_+)$;
- $(\hat{H}_4)$ $f \in C^1(\mathbb{R})$ and $f(0) = 0$;
- $(\hat{H}_5)$ $F \in C^1([\rho,1] \times \mathbb{R}_+)$.

Then, we have the following theorem.

Theorem 3.3. *Let (H_1), $(\hat{H}_2) - (\hat{H}_5)$ hold and $\lambda > 0$. Then there exists a unique local weak solution u of Prob. (3.11) such that*

$$\begin{aligned} &u \in L^\infty(0,T; H^2 \cap H_0^1) \cap C^0([0,T]; H_0^1) \cap C^1([0,T]; L^2), \\ &u' \in C^0([0,T]; L^2) \cap L^\infty(0,T; H_0^1), \\ &u'' \in L^\infty(0,T; L^2), \end{aligned} \tag{3.13}$$

with $T > 0$ small enough.

In what follows, we prove that if $\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)dz - p\int_\rho^1 xdx\int_0^{\tilde{u}_0(x)} f(z)dz > 0$ and the initial energy $E(0) = \frac{1}{2}\|\tilde{u}_1\|_0^2 + \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)\,dz - \int_\rho^1 xdx\int_0^{\tilde{u}_0(x)} f(z)dz$ and F are small enough, then any global weak solution is general decay as $t \to +\infty$. For this purpose, we strengthen the following assumptions

$(\bar{H}_3)$ $f \in C^1(\mathbb{R})$, $f(0)=0$ and there exist constants α, $\beta > 2$; d_1, $\overline{d}_1 > 0$, such that
(i) $yf(y) \le d_1 \int_0^y f(z)dz$, for all $y \in \mathbb{R}$,
(ii) $\int_0^y f(z)dz \le \overline{d}_1 \left(|y|^\alpha + |y|^\beta\right)$, for all $y \in \mathbb{R}$;

$(\bar{H}_4)$ $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that
(i) $L_* = \mu_* - \overline{g}(\infty) > 0$,
(ii) There exists a function $\xi \in C^1(\mathbb{R}_+)$ such that
(j) $\xi'(t) \le 0 < \xi(t)$, $\forall t \ge 0$, $\int_0^\infty \xi(t)\,dt = \infty$,
(jj) $g'(t) \le -\xi(t)\,g(t) < 0$, $\forall t \ge 0$,
with $\overline{g}(t) = \int_0^t g(s)\,ds$ and $\overline{g}(\infty) = \int_0^\infty g(s)\,ds$;

$(\bar{H}_5)$ $F \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$ and there exist positive constants C_0, γ_0 such that $\|F(t)\|_0^2 \le C_0 \exp(-\gamma_0 t)$, $\forall t \ge 0$;

$(\bar{H}_6)$ $p > d_1$.

We first construct the Lyapunov functional in the form

$$\mathcal{L}(t) = E(t) + \delta\Psi(t), \tag{3.14}$$

where $\delta > 0$ is chosen suitably and

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}(g*u)(t) - \int_\rho^1 x dx \int_0^{u(x,t)} f(z)dz \\ &\quad + \frac{1}{2}\left(\int_0^{\|u_x(t)\|_0^2} \mu(z)\,dz - \overline{g}(t)\|u_x(t)\|_0^2\right) \\ &= \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{p}I(t) \\ &\quad + \left(\frac{1}{2} - \frac{1}{p}\right)\left((g*u)(t) + \int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \overline{g}(t)\|u_x(t)\|_0^2\right), \end{aligned} \tag{3.15}$$

$$\Psi(t) = \langle u(t), u'(t)\rangle + \frac{\lambda}{2}\|u(t)\|_0^2, \tag{3.16}$$

with

$$I(t) = (g*u)(t) + \int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \overline{g}(t)\|u_x(t)\|_0^2 - p\int_\rho^1 x dx \int_0^{u(x,t)} f(z)dz, \tag{3.17}$$

and

$$(g*u)(t) = \int_0^t g(t-s)\|u_x(t) - u_x(s)\|^2 ds, \;\; \overline{g}(t) = \int_0^t g(s)ds. \tag{3.18}$$

We have the following estimate for $E'(t)$.

Lemma 3.4. *For all $\varepsilon_1 > 0$, we have*

$$\begin{aligned} &(i) \quad E'(t) \leq \tfrac{1}{2}\|F(t)\|_0 + \tfrac{1}{2}\|F(t)\|_0 \|u'(t)\|_0^2, \\ &(ii) \quad E'(t) \leq -(\lambda - \frac{\varepsilon_1}{2})\|u'(t)\|_0^2 - \frac{1}{2}\xi(t)(g*u)(t) + \frac{1}{2\varepsilon_1}\|F(t)\|_0^2. \end{aligned} \tag{3.19}$$

Proof. Multiplying $(3.11)_1$ by $xu'(x,t)$, and integrating over $[\rho, 1]$, we get

$$E'(t) = -\lambda\|u'(t)\|_0^2 + \frac{1}{2}(g' * u)(t) - \frac{1}{2}g(t)\|u_x(t)\|_0^2 + \langle F(t), u'(t)\rangle. \tag{3.20}$$

Using assumptions $(\bar{H}_4)$, we obtain $\frac{1}{2}(g' * u)(t) \leq -\xi(t)(g*u)(t)$, so

$$E'(t) \leq \langle F(t), u'(t)\rangle \leq \frac{1}{2}\|F(t)\|_0 + \frac{1}{2}\|F(t)\|_0\|u'(t)\|_0^2.$$

By applying Cauchy-Schwartz inequality, we have

$$\langle F(t), u'(t)\rangle \leq \frac{1}{2\varepsilon_1}\|F(t)\|_0^2 + \frac{\varepsilon_1}{2}\|u'(t)\|_0^2, \ \forall \varepsilon_1 > 0. \tag{3.21}$$

It is easy to see that $(3.19)_{ii}$ holds from (3.20) and (3.21). Lemma 3.4 is proven. □

Lemma 3.5. *Let $(\hat{H}_2)$, $(\bar{H}_3) - (\bar{H}_6)$ hold and $(\tilde{u}_0, \tilde{u}_1) \in (H_0^1 \cap H^2) \times H_0^1$ such that $I(0) > 0$ and*

$$\eta^* \equiv L_* - p\bar{d}_1(1-\rho)\left[\left(\frac{1-\rho}{\rho}\right)^{\alpha/2} R_*^{\alpha-2} + \left(\frac{1-\rho}{\rho}\right)^{\beta/2} R_*^{\beta-2}\right] > \mu_{\max} - \frac{p}{d_1}L_*, \tag{3.22}$$

where

$$\begin{aligned} &R_* = \left(\frac{2pE_*}{(p-2)L_*}\right)^{1/2}, E_* = \left(E(0 + \frac{1}{2}\varrho_1\right)\exp(\varrho_1), \\ &\varrho_1 = \int_0^\infty \|F(t)\|\,dt, \ L_* = \mu_* - \overline{g}(\infty), \ \mu_{\max} = \max_{0\leq z\leq R_*^2} \mu(z), \end{aligned}$$

Then $I(t) > 0$ for all $t \geq 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $\tilde{T}_1 > 0$ such that

$$I(t) = I(u(t)) > 0, \ \forall t \in [0, \tilde{T}_1], \tag{3.23}$$

we get

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g*u)(t) + \int_0^{\|u_x(t)\|_0^2}\mu(z)dz - \overline{g}(t)\|u_x(t)\|_0^2\right] \\ &\geq \frac{1}{2}\|u'(t)\|_0^2 + \frac{(p-2)}{2p}\left[(g*u)(t) + L_*\|u_x(t)\|_0^2\right], \ \forall t \in [0, \tilde{T}_1]. \end{aligned} \tag{3.24}$$

Combining $(3.19)_i$, (3.24) and using Gronwall's inequality, we obtain

$$\|u_x(t)\|_0^2 \leq \frac{2p}{(p-2)L_*}E(t) \leq \frac{2pE_*}{(p-2)L_*} \equiv R_*^2, \ \forall t \in [0, \tilde{T}_1]. \tag{3.25}$$

Then it follows from $(\overline{H}_3)(ii)$ that

$$\begin{aligned} & p\int_\rho^1 xdx\int_0^{u(x,t)} f(z)dz \\ & \leq p\bar{d}_1\left(\|u(t)\|_{L^\alpha}^\alpha+\|u(t)\|_{L^\beta}^\beta\right) \\ & \leq p\bar{d}_1\left(1-\rho\right)\left(\sqrt{\left(\frac{1-\rho}{\rho}\right)^\alpha}\|u_x(t)\|_0^\alpha+\sqrt{\left(\frac{1-\rho}{\rho}\right)^\beta}\|u_x(t)\|_0^\beta\right) \\ & \leq p\bar{d}_1\left(1-\rho\right)\left[\sqrt{\left(\frac{1-\rho}{\rho}\right)^\alpha}R_*^{\alpha-2}+\sqrt{\left(\frac{1-\rho}{\rho}\right)^\beta}R_*^{\beta-2}\right]\|u_x(t)\|_0^2. \end{aligned} \tag{3.26}$$

Thus, $I(t)\geq\eta^*\|u_x(t)\|_0^2+(g*u)(t)>0$, $\forall t\in[0,\tilde{T}_1]$, where the positive constant η^* is defined as in (3.22).

Put $T_\infty=\sup\{T>0: I(t)>0, \forall t\in[0,T]\}$. If $T_\infty<+\infty$, then by the continuity of $I(t)$, we have $I(T_\infty)\geq 0$. If $I(T_\infty)>0$, by the same arguments as above, we can deduce that there exists $\tilde{T}_\infty>T_\infty$ such that $I(t)>0$, $\forall t\in[0,\tilde{T}_\infty]$. This is contrary to the definition of T_∞, so we get $I(t)>0$, $\forall t\geq 0$.

If $I(T_\infty)=0$, then

$$0=I(T_\infty)\geq\eta^*\|u_x(T_\infty)\|_0^2+(g*u)(T_\infty)\geq 0.$$

Therefore

$$u_x(T_\infty)=(g*u)(T_\infty)=0.$$

By the fact that the functions $s\mapsto g(T_\infty-s)\|u_x(s)-u_x(T_\infty)\|_0^2$ is continuous on $[0,T_\infty]$ and $g(T_\infty-s)>0$, for all $s\in[0,T_\infty]$, and

$$(g*u)(T_\infty)=\int_0^{T_\infty} g(T_\infty-s)\|u_x(s)\|_0^2 ds=0,$$

we have $\|u_x(s)\|_0=0$, for all $s\in[0,T_\infty]$. Thus, $u(0)=0$. This is contrary to $I(0)>0$. Consequently, $T_\infty=+\infty$, i.e. $I(t)>0$ for all $t\geq 0$. Lemma (3.5) is proved. □

Next, we put

$$E_1(t)=\|u'(t)\|_0^2+\|u_x(t)\|_0^2+(g*u)(t)+I(t). \tag{3.27}$$

Then, we have the following Lemma.

Lemma 3.6. *Let the assumptions of Lemma 3.5 hold. Then, there exist positive constants β_1, $\overline{\beta}_1$, β_2, $\overline{\beta}_2$ such that*

$$\begin{aligned} \beta_1 E_1(t)\leq\mathcal{L}(t)\leq\beta_2 E_1(t),\ \forall t\geq 0, \\ \overline{\beta}_1 E_1(t)\leq E(t)\leq\overline{\beta}_2 E_1(t),\ \forall t\geq 0. \end{aligned}$$

Proof. Lemma (3.6) is proved by using some simple estimates, hence we omit the details. □

Lemma 3.7. *For all $\varepsilon_2>0$, we have*

$$\begin{aligned} \Psi'(t)\leq\|u'(t)\|^2+\left(\frac{d_1\delta_1}{p}+\frac{1}{2\varepsilon_2}\right)(g*u)(t)+\frac{1}{2\varepsilon_2}\|F(t)\|_0^2-\frac{d_1\delta_1}{p}I(t) \\ -\left(\mu_*+\eta^*\frac{d_1}{p}(1-\delta_1)-\left(1+\frac{\varepsilon_2}{2}-\frac{d_1}{p}\right)\overline{g}(\infty)-\frac{\varepsilon_2(1-\rho)^2}{4\rho}-\frac{d_1}{p}\mu_{\max}\right)\|u_x(t)\|_0^2. \end{aligned} \tag{3.28}$$

Proof. Multiplying $(3.11)_1$ by $xu(x,t)$, and integrating over $[\rho, 1]$, we get

$$\Psi'(t) = \|u'(t)\|_0^2 - \mu\left(\|u_x(t)\|_0^2\right)\|u_x(t)\|_0^2 + \langle f(u(t)), u(t)\rangle + \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle\, ds + \langle F(t), u(t)\rangle. \tag{3.29}$$

From the following inequalities

$$\begin{aligned} -\mu\left(\|u_x(t)\|^2\right)\|u_x(t)\|_0^2 &\leq -\mu_* \|u_x(t)\|_0^2, \\ \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle\, ds &\leq \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(t)\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}(g*u)(t), \\ \langle f(u(t)), u(t)\rangle &\leq \frac{d_1}{p}\left[\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + (g*u)(t) - \overline{g}(t)\|u_x(t)\|_0^2 - I(t)\right], \\ I(t) &\geq (g*u)(t) + \eta^*\|u_x(t)\|_0^2, \\ \langle F(t), u(t)\rangle &\leq \frac{\varepsilon_2(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2,\ \forall \varepsilon_2 > 0, \end{aligned} \tag{3.30}$$

we have

$$\begin{aligned} \Psi'(t) &\leq \|u'(t)\|_0^2 - \mu_*\|u_x(t)\|_0^2 + \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(t)\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}(g*u)(t) \\ &\quad + \frac{d_1}{p}\left[\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + (g*u)(t) - \overline{g}(t)\|u_x(t)\|_0^2 - I(t)\right] \\ &\quad + \frac{\varepsilon_2(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 \\ &\leq \|u'(t)\|_0^2 + \left(\frac{d_1}{p} + \frac{1}{2\varepsilon_2}\right)(g*u)(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 - \frac{d_1\delta_1}{p}I(t) - \frac{d_1(1-\delta_1)}{p}I(t) \\ &\quad - \left(\mu_* - \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(\infty) - \frac{\varepsilon_2(1-\rho)^2}{4\rho} - \frac{d_1}{p}\mu_{\max}\right)\|u_x(t)\|_0^2 \\ &\leq \|u'(t)\|^2 + \left(\frac{d_1\delta_1}{p} + \frac{1}{2\varepsilon_2}\right)(g*u)(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 - \frac{d_1\delta_1}{p}I(t) \\ &\quad - \left(\mu_* + \eta^*\frac{d_1}{p}(1-\delta_1) - \left(1+\frac{\varepsilon_2}{2} - \frac{d_1}{p}\right)\overline{g}(\infty) - \frac{\varepsilon_2(1-\rho)^2}{4\rho} - \frac{d_1}{p}\mu_{\max}\right)\|u_x(t)\|_0^2. \end{aligned}$$

Lemma (3.7) is proved completely. □ □

Putting $\rho(t) = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|_0^2$, from Lemma 3.4–Lemma 3.7, we obtain

$$\begin{aligned} \mathcal{L}'(t) &\leq -(\lambda - \frac{\varepsilon_1}{2} - \delta)\|u'(t)\|_0^2 - \frac{1}{2}\xi(t)(g*u)(t) + \rho(t) \\ &\quad + \delta\left(\frac{d_1\delta_1}{p} + \frac{1}{2\varepsilon_2}\right)(g*u)(t) - \frac{\delta d_1\delta_1}{p}I(t) \\ &\quad - \delta\left(\mu_* + \eta^*\frac{d_1}{p}(1-\delta_1) - \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(\infty) - \frac{\varepsilon_2(1-\rho)^2}{4\rho} - \frac{d_1}{p}\mu_{\max}\right)\|u_x(t)\|_0^2. \end{aligned} \tag{3.31}$$

We have

$$\lim_{\substack{\delta_1\to 0_+\\ \varepsilon_2\to 0_+}} \left(\mu_* + \eta^* \frac{d_1}{p}(1-\delta_1) - \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(\infty) - \frac{\varepsilon_2(1-\rho)^2}{4\rho} - \frac{d_1}{p}\mu_{\max}\right)$$
$$= \mu_* + \eta^* \frac{d_1}{p} - \overline{g}(\infty) - \frac{d_1}{p}\mu_{\max} > 0.$$

Then, we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ small enough such that

$$\theta_1(\delta_1,\varepsilon_2) \equiv \mu_* + \eta^* \frac{d_1}{p}(1-\delta_1) - \left(1+\frac{\varepsilon_2}{2}\right)\overline{g}(\infty) - \frac{\varepsilon_2(1-\rho)^2}{4\rho} - \frac{d_1}{p}\mu_{\max} > 0.$$

Moreover, we also choose $\delta > 0$, $\varepsilon_1 > 0$ small enough and satisfying

$$\theta_2(\varepsilon_1,\delta) = \lambda - \frac{\varepsilon_1}{2} - \delta > 0.$$

Putting

$$\theta_* = \min\left\{\delta\theta_1, \delta\theta_2, \frac{\delta d_1\delta_1}{p}\right\},\ \theta_3 = \delta\left(\frac{d_1\delta_1}{p} + \frac{1}{2\varepsilon_2}\right),$$

we get that

$$\mathcal{L}'(t) \leq -\theta_* E_1(t) + (\theta_* + \theta_3)(g*u)(t) + \rho(t). \tag{3.32}$$

Combining $(3.19)_{ii}$ and (3.32), we obtain

$$\begin{aligned}\xi(t)\mathcal{L}'(t) &\leq -\theta_*\xi(t)E_1(t) + (\theta_*+\theta_3)\xi(t)(g*u)(t) + \xi(0)\rho(t)\\ &\leq -\theta_*\xi(t)E_1(t) + 2(\theta_*+\theta_3)\left[-E'(t) + \frac{1}{2\varepsilon_1}\|F(t)\|_0^2\right] + \xi(0)\rho(t)\\ &\leq -\theta_*\xi(t)E_1(t) - 2(\theta_*+\theta_3)E'(t) + \overline{C}_0\exp(-\gamma_0 t),\end{aligned}$$

where $\overline{C}_0 = \left[\frac{\theta_*+\theta_3}{\varepsilon_1} + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\right] C_0$.
Setting the functional $L(t) = \xi(t)\mathcal{L}(t) + 2(\theta_*+\theta_3)E(t)$, we have

$$\begin{aligned}L(t) &\leq \left[\xi(0)\beta_2 + 2(\theta_*+\theta_3)\overline{\beta}_2\right] E_1(t) \equiv \widehat{\beta}_2 E_1(t),\\ L'(t) &\leq \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + 2(\theta_*+\theta_3)E'(t)\\ &\leq -\theta_*\xi(t)E_1(t) + \overline{C}_0\exp(-\gamma_0 t)\\ &\leq -\frac{\theta_*}{\widehat{\beta}_2}\xi(t)L(t) + \overline{C}_0\exp(-\gamma_0 t).\end{aligned} \tag{3.33}$$

By choosing $0 < \overline{\gamma} < \min\left\{\frac{\theta_*}{\widehat{\beta}_2}, \frac{\gamma_0}{\xi(0)}\right\}$, we get

$$L'(t) + \overline{\gamma}\xi(t)L(t) \leq \overline{C}_0\exp(-\gamma_0 t). \tag{3.34}$$

It leads to

$$L(t) \leq \left(L(0) + \frac{\overline{C}_0}{\gamma_0 - \overline{\gamma}\xi(0)}\right)\exp\left(-\overline{\gamma}\int_0^t \xi(s)ds\right). \tag{3.35}$$

On the other hand, we also have

$$L(t) = \xi(t)\mathcal{L}(t) + 2(\theta_* + \theta_3)E(t) \geq 2(\theta_* + \theta_3)\overline{\beta}_1 E_1(t) \geq 2(\theta_* + \theta_3)\overline{\beta}_1 \left(\|u'(t)\|_0^2 + \|u_x(t)\|_0^2\right). \tag{3.36}$$

Therefore, we obtain the main result in this section as follows.

Theorem 3.8. *Let the assumptions of Lemma 3.5 hold. Then, any global weak solution of Prob. (3.11) is general decay as $t \to +\infty$. Moreover, there exist positive constants $\bar{C}$, $\bar{\gamma}$, such that*

$$\|u'(t)\|_0^2 + \|u_x(t)\|_0^2 \leq \bar{C}\exp\left(-\overline{\gamma}\int_0^t \xi(s)ds\right), \quad \forall t \geq 0. \tag{3.37}$$

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