ĐẠI HỌC THỦ DẦU MỘT
2009 THU DAU MOT UNIVERSITY

Thu Dau Mot University
Journal of Science
ISSN 2615 - 9635
journal homepage: ejs.tdmu.edu.vn

Global solutions to a time-fractional Cauchy problem

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Article Info: *Received Feb. 22nd, 2023, Accepted May 15th, 2023, Available online June 15th, 2023*

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https://doi.org/10.37550/tdmu.EJS/2023.05.419

ABSTRACT

In the current work, we study a Cauchy problem for a time-fractional pseudo-parabolic equation with a globally Lipschitz source term. We prove the unique existence of a mild solution to the problem, by the common Banach fixed point theorem. This solution is then verified that exists globally in time by Grönwall's inequality. Compare to previous works about the similar issuse, we approach in a way that does not require using weighted spaces. Although our approach share a similar spirit to previous studies, our method seems to be more precise and natural.

Keywords: *Global solutions, time-fractional Cauchy problem, pseudo-parabolic equation*

1 Introduction

In this paper, we investigate the following Cauchy problem

$$\begin{cases} {}_C\mathrm{D}_t^\alpha \left(I-\Delta\right) u(x,t) = \Delta u(x,t) + K(u(x,t)) & (x,t)\in\mathcal{D}\times\mathbb{R}^+, \\ u(x,t)=0, & (x,t)\in\partial\mathcal{D}\times\mathbb{R}^+, \\ u(x,t)=u_0(x), & (x,t)\in\mathcal{D}\times\{0\}, \end{cases} \tag{1.1}$$

where $\mathcal{D}$ is a smooth bounded domain of $\mathbb{R}^N$, $N\geqslant 1$, u_0 is the initial data and ${}_C\mathrm{D}_t^\alpha$ is the Caputo derivative of order $\alpha\in(0,1)$, defined by

$${}_C\mathrm{D}_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\zeta)^{-\alpha}\partial_\zeta u(\zeta)\mathrm{d}\zeta,$$

where $u \in C^1([0,\infty))$. Recently, fractional PDEs have attracted the interest of many mathematicians due to their useful features in practical models. In fact, capturing memory effects on diffusion processes leads to the modified diffusion equation

$$ {}_C\mathrm{D}_t^\alpha u = \Delta u. $$

We refer the reader to the following works [2,3,5,8] for beatiful results of existence and uniqueness of mild solutions. In view of pseudo-parabolic models, time-fractional derivatives plays a same role as in the parabolic case. The standard works on this topic are [1,6,7].

Another interesting point of Problem (1.1) comes from the affects of F on the solution. In this study, we suppose that F satisfies the following Lipschitz hypotheses:

$$ \left\|F(u) - F(v)\right\|_{L^2} = \mathscr{O}\left(\left\|(-\Delta)^s(u-v)\right\|_{L^2}\right), \tag{1.2} $$

$$ \left\|F(u)\right\|_{L^2} = \mathscr{O}\left(\left\|(-\Delta)^s u\right\|_{L^2}\right), \tag{1.3} $$

where $s \in (0,1)$, provided $\|(-\Delta)^s u\|_{L^2}$ and $\|(-\Delta)^s v\|_{L^2}$ are finite. We note that another global Lipschitz type of F for (1.1) was investigated in [6]. In this paper, Tuan et.al. used a weighted norm in a Banach space to get the global existence and uniqueness of a mild solution. In this paper, we derive the same result by another approach in which a blow-up criterion for local solutions is provided. Then, a Grönwall type inequality helps us to get the global results. Here we note that the local existence and uniqueness of a mild solution were proved by the Banach fixed point theorem. Since the global existence is provided by fundamental properties of the local solution, our approach in this work seems to be more natural than in [6]. Accordingly to this advance, this approach can be applied further in other fractional models such as fractional chemotaxis system, fractional Fisher equation, etc.

The paper are outlined as follows. Basic settings about function spaces and linear estimates are provided in Secntion 2. We prove main results about global existence and uniqueness of mild solutions in Section 3.

2 Settings

Through this work, we write $A = \mathscr{O}(B)$ for the inequality $|A| \leqslant C|B|$ where C is a positive constant whose value can change line by line. In addition, we shorten $\Lambda := -\Delta$.

We define Hilbert spaces via the spectral problem of Λ. More precisely, we denote by $\{\lambda_k\}_{k\in\mathbb{N}}, \{e_k\}_{k\in\mathbb{N}}$ the set of eigenvalues and eigenvectors associated to Λ in $L^2(\mathcal{D})$, respectively. Then, we define the followinf Hilbert space

$$ \mathbb{X}^s(\mathcal{D}) := \left\{u \in L^2(\mathcal{D}) \;\middle|\; \left\|\Lambda^s u\right\|_{L^2} < \infty\right\}. $$

Before moving to fractional settings, we first introduce the following notation which make our definition more rigorous.

$$ \delta_j(\alpha, A, B) := \left(\frac{-BA^\alpha}{1+B}\right)^j, $$

where $A \geqslant,\ B > 0$ and $j \in \mathbb{N} \cup \{0\}$. We are now ready to define the definition of mild solutions to Problem (1.1).

Definition 2.1 (See [6]). *Let $u_0 \in L^2(\mathcal{D})$. A function u is called a mild solution to Problem (1.1) if it satisfies the following integral equality*

$$\mathscr{S}[u](x,t) = u(x,t) := M_1(t)u_0(x) + \int_0^t M_2(t-\zeta)K(u(x,\zeta))\mathrm{d}\zeta,$$

where we define for $t > 0$

$$M_1(t)u := \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{\delta_j(\alpha,t,\lambda_k)u}{\Gamma(\alpha j+1)} e_k,$$
$$M_2(t)u := \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{t^{\alpha-1}\delta_j(\alpha,t,\lambda_k)u}{(1+\lambda_k)\Gamma(\alpha j+\alpha)} e_k.$$

Lemma 2.2 (See [6]). *Let $\alpha, s \in (0,1)$ and $u \in \mathbb{X}(\mathcal{D})$. Then, the following estimates hold*

- $\|\Lambda^s M_1(t)u\|_{L^2} = \mathscr{O}\left(\|\Lambda^s u\|_{L^2}\right)$ *for any $t > 0$.*
- $\|\Lambda^s M_2(t)u\|_{L^2} = \mathscr{O}\left(t^{\alpha-1}\|u\|_{L^2}\right)$ *for any $t > 0$.*

3 Existence and uniqueness

Theorem 3.1. *Let $u_0 \in \mathbb{X}(\mathcal{D})$. Then, there exists a positive constant $T > 0$ and a unique mild solution to Problem (1.1) in $C([0,T];\mathbb{X}^s(\mathcal{D}))$.*

Proof. The proof begins by defining the following space for a fixed constant $R = 2\|\Lambda^s u_0\|_{L^2}$

$$E_T := \left\{ u \in C\left([0,T]; X^s(\mathcal{D})\right) \;\middle|\; \sup_{[0,T]} \|\Lambda^s u(t)\| < R \right\},$$

for some $T > 0$. We aim to show that $\mathscr{S}$ is a contraction mapping from E_T to E_T. To this end, we present the proof via the following claims

Claim 1: $\mathscr{S}$ is E_T invariant.
For $u_0 \in \mathbb{X}^s(\mathcal{D})$, we get the following estimate

$$\|\Lambda^s M_1(t)u_0\|_{L^2} = \mathcal{O}\left(\|\Lambda^s u_0\|_{L^2}\right), \quad \text{for any } t \in [0,T]. \tag{3.1}$$

Let u be an arbitary element of E_T. We can find the following estimate

$$\begin{aligned}\left\|\Lambda^s\left(\mathscr{S}[u](t) - M_1(t)u_0\right)\right\|_{L^2} &= \mathcal{O}\left(\int_0^t \left\|\Lambda^s M_2(t-\zeta)K(u(\zeta))\right\|_{L^2}\mathrm{d}\zeta\right)\\ &= \mathcal{O}\left(\int_0^t (t-\zeta)^{\alpha-1}\left\|K(u(\zeta))\right\|_{L^2}\mathrm{d}\zeta\right),\end{aligned}$$

where we have used Lemma 2.2. By hypothesis for K, the following estimate holds

$$\left\|K(u(t))\right\|_{L^2} = \mathcal{O}\left(\|\Lambda^s u(t)\|_{L^2}\right), \quad \text{for any } t \in [0,T].$$

The latter two estimates then imply

$$\begin{aligned}\left\|\Lambda^s\left(\mathscr{S}[u](t)-M_1(t)u_0\right)\right\|_{L^2}&=\mathcal{O}\left(\int_0^t(t-\zeta)^{\alpha-1}\left\|\Lambda^s u(\zeta)\right\|_{L^2}\mathrm{d}\zeta\right)\\&=\mathcal{O}(t^\alpha)\sup_{[0,T]}\left\|\Lambda^s u(t)\right\|_{L^2}\\&=\mathcal{O}(t^\alpha)R\end{aligned}$$

Then, we can choose a sufficiently small T such that the right-hand of then above inequality is less than $R/2$.

In view of all above results, one can find a positive constant T such that for any $u\in E_T$, $\mathscr{S}[u]$ also belongs to E_T. More precisely, there holds

$$\begin{aligned}\sup_{[0,T]}\left\|\Lambda^s\mathscr{S}[u](t)\right\|_{L^2}&\leqslant\sup_{[0,T]}\left\|\Lambda^s M(1)u_0\right\|_{L^2}+\sup_{[0,T]}\left\|\Lambda^s\left(\mathscr{S}[u](t)-M_1(t)u_0\right)\right\|_{L^2}\\&\leqslant R,\quad\text{for any } u\in E_T.\end{aligned}$$

Claim 2: $\mathscr{S}:E_T\to E_T$ is a contraction mapping.
Taking $u,v\in E_T$, it is obvious that

$$\left\|\Lambda^s\left(\mathscr{S}[u](t)-\mathscr{S}[v](t)\right)\right\|_{L^2}=\mathscr{O}\left(\int_0^t\left\|\Lambda^s M_2(t-\zeta)\left(K(u(\zeta))-K(v(\zeta))\right)\right\|_{L^2}\mathrm{d}\zeta\right).$$

Then, by Lemma 2.2 we can find that

$$\left\|\Lambda^s\left(\mathscr{S}[u](t)-\mathscr{S}[v](t)\right)\right\|_{L^2}=\mathscr{O}\left(\int_0^t(t-\zeta)^{\alpha-1}\left\|\left(K(u(\zeta))-K(v(\zeta))\right)\right\|_{L^2}\mathrm{d}\zeta\right).$$

The hypothesis (1.2) yields

$$\begin{aligned}\left\|\Lambda^s\left(\mathscr{S}[u](t)-\mathscr{S}[v](t)\right)\right\|_{L^2}&=\mathscr{O}\left(\int_0^t(t-\zeta)^{\alpha-1}\left\|\Lambda^s\left(u(\zeta)-v(\zeta)\right)\right\|_{L^2}\mathrm{d}\zeta\right)\\&=\mathscr{O}(t^\alpha)\sup_{[0,T]}\left\|\Lambda^s\left(u(t)-v(t)\right)\right\|_{L^2}.\end{aligned}$$

Choosing T approriately small outputs

$$\left\|\Lambda^s\left(\mathscr{S}[u](t)-\mathscr{S}[v](t)\right)\right\|_{L^2}=\mathscr{O}(\theta)\sup_{[0,T]}\left\|\Lambda^s\left(u(t)-v(t)\right)\right\|_{L^2},$$

for all $t\in[0,T]$, where $\theta\in(0,1)$. Consequently, one has

$$\sup_{[0,T]}\left\|\Lambda^s\left(\mathscr{S}[u](t)-\mathscr{S}[v](t)\right)\right\|_{L^2}=\mathscr{O}(\theta)\sup_{[0,T]}\left\|\Lambda^s\left(u(t)-v(t)\right)\right\|_{L^2},$$

this implies that $\mathscr{S}$ is a contraction from E_T to E_T.

Accordingly to the above claims, the Banach fixed point theorem can be applied to deduce the unique existence of a mild solution to Problem (1.1), provided that E_T is a complete metric space with respect to the metric

$$d_{E_T}(u,v):=\sup_{[0,T]}\left\|\Lambda^s\left(u(t)-v(t)\right)\right\|_{L^2},$$

for any $u,v\in E_T$. The proof is thus completed. □

Theorem 3.2. *Let $u_0 \in \mathbb{X}(\mathcal{D})$. Then, the solution u in Theorem 3.1 exists globally in time.*
Proof.
Part 1: Extension of a local mild solution.
Suppose that u is a mild solution to Problem (1.1) in $C([0,T];\mathbb{X}^s(\mathcal{D}))$ for some $T > 0$. We first note that due to the continuity of u, one can extend u to some spaces of larger time intervals. In fact, for a fixed constant R' we consider the following function space

$$\widetilde{E}_{T+\varepsilon,R} = \left\{ w \in C([0,T+\varepsilon];\mathbb{X}^s(\mathcal{D})) \;\middle|\; w_{[0,T]} \equiv u \text{ and } \sup_{[T,T+\varepsilon]} \left\| (w(t) - u(T)) \right\|_{L^2} \leqslant R \right\},$$

for some $\varepsilon > 0$. This space is a metric space with the metric generated as follows

$$d_{\widetilde{E}_{T+\varepsilon,R}(w,v)} := \sup_{[0,T+\varepsilon]} \left\| \Lambda^s (w(t) - v(t)) \right\|_{L^2},$$

for any $w, v \in \widetilde{E}_{T+\varepsilon,R}$. For $w \in \widetilde{E}_{T+\varepsilon,R}$, by the continuity of $\mathscr{S}[w](t)$ we can easily prove that $\mathscr{S}[w] \in \widetilde{E}_{T+\varepsilon,R}$. In addition, by similar arguments as in the previous theorem we can also verify that $\mathscr{S}$ is a contraction mapping on $\widetilde{E}_{T+\varepsilon,R}$. Thus, the extension of u is ensured.

Part 2: Blow-up criterion.
We first define

$$T_{\max} := \sup \left\{ T > 0 \;\middle|\; \text{Problem (1.1) possesses a unique mild solution } u \text{ on } [0,T) \right\}.$$

We aim to show a criterion that if $T_{\max} < \infty$

$$\lim_{t \to T_{\max}^-} \left\| \Lambda^s u(t) \right\|_{L^2} = \infty.$$

Suppose by contraction that $T_{\max} < \infty$ and

$$\lim_{t \to T_{\max}^-} \left\| \Lambda^s u(t) \right\|_{L^2} < \infty.$$

Let $\{t_l\}_{l \in \mathbb{N}}$ be a subset of $[0, T_{\max})$ such that $t_l \longrightarrow T_{\max}$ as $l \to \infty$. Obviously, $\{t_l\}_{l \in \mathbb{N}}$ is also a Cauchy sequence. Then, for any $\epsilon > 0$ there exists a constant l_0 such that

$$\left| t_{l_1} - t_{l_2} \right| < \epsilon, \quad \text{for all } l_1, l_2 \geqslant l_0.$$

In view of this observation and the continuity of u, for any $\epsilon > 0$ we can find a sufficiently large constant l_0 such that

$$\left\| \Lambda^s (u(t_{l_1}) - u(t_{l_2})) \right\|_{L^2} \leqslant \epsilon, \quad \text{for all } l_1, l_2 \geqslant l_0.$$

Therefore, one can conclude that $\{u(t_l)\}_{l \in \mathbb{N}}$ is also a Cauchy sequence. Then, the completeness of $C([0, T_{\max}); \mathbb{X}^s(\mathcal{D}))$ implies the existence of a limit $\widetilde{u}$ such that

$$\widetilde{u} := \lim_{l \to \infty} u(t_l) = u(T_{\max}),$$

by the dominated convergence theorem. This implies the determination of u at $t = T_{\max}$. By Part 1, we can extend u to some spaces of larger time interval. This contradicts the definition of $T_{\max}$ and completes our second part.

Part 3. Applying Grönwall's inequality. We consider the graph norm of u at any time $t \in [0, T_{\max})$ as follows

$$\left\|\Lambda^s u(t)\right\|_{L^2} \leqslant \left\|\Lambda^s M_1(t) u_0\right\|_{L^2} + \int_0^t \left\|\Lambda^s M_2(t-\zeta) G(u(\zeta))\right\|_{L^2} \mathrm{d}\zeta.$$

Estimate in a same way as in Theorem 3.1, one can easily deduce

$$\left\|\Lambda^s u(t)\right\|_{L^2} \leqslant \left\|\Lambda^s u_0\right\|_{L^2} + \int_0^t (t-\zeta)^{\alpha-1} \left\|\Lambda^s u(\zeta)\right\|_{L^2} \mathrm{d}\zeta, \quad \text{for all } t \in [0, T_{\max}).$$

Applying Grönwall's inequality (See [4]) yields

$$\left\|\Lambda^s u(t)\right\|_{L^2} = \mathscr{O}\left(M(t)\right) < \infty,$$

for all $t < \infty$. By Part 2, we conclude $T_{\max} = \infty$. And the proof is completed. □

Acknowledgement The author Nguyen Anh Tuan would like to thank the support from Van Lang University.

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