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The Dirichlet problem for a wave equation of Kirch-hoff-Carrier type with a nonlinear viscoelastic term

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ABSTRACT

In this paper, we consider the Dirichlet problem for a wave equation of Kirchhoff-Carrier type with a nonlinear viscoelastic term. It consists of two main parts. In Part 1, we establish existence and uniqueness of a weak solution by applying the Faedo-Galerkin method and the standard arguments of density corresponding to the regularity of initial conditions. In Part 2, we give a sufficient condition for the global existence and exponential decay of the weak solutions by defining a modified energy functional together with the technique of Lyapunov functional.

Keywords: *Faedo-Galerkin method, Nonlinear Kirchhoff-Carrier wave equation, local existence, global existence, exponential decay.*

1 Introduction

In this paper, we consider the Dirichlet problem for a wave equation of Kirchhoff-Carrier type as follows

$$\begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial}{\partial x}\left[\mu_1\left(x,t,u\left(x,t\right),\|u(t)\|^2,\|u_x(t)\|^2\right)u_x\right] \\ \qquad\qquad + \displaystyle\int_0^t \mu_2\left(t,s,\|u(s)\|^2,\|u_x(s)\|^2\right)u_{xx}(x,s)ds \\ \qquad\qquad = f\left(x,t,u,u_x,u_t\right),\ 0<x<1,\ 0<t<T, \\ u(0,t)=u(1,t)=0, \\ u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is given constant; μ_1, μ_2, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions, which will be specified later; the nonlinear terms $\mu_1\left(x,t,u\left(x,t\right),\|u(t)\|^2,\|u_x(t)\|^2\right)$ and $\mu_2\left(t,s,\|u(s)\|^2,\|u_x(s)\|^2\right)$ in Eq. $(1.1)_1$ depend on the integrals

$$\|u_x(t)\|^2 = \int_0^1 u_x^2\left(x,t\right)dx,\quad \|u(t)\|^2 = \int_0^1 u^2\left(x,t\right)dx. \tag{1.2}$$

This problem has the so-called model of Kirchhoff - Carrier type because it connects Kirchhoff and Carrier equations. For more details, Eq. $(1.1)_1$ has its origin in the nonlinear vibration of an elastic string (Kirchhoff [4]), for which the associated equation is

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L} \int_0^L \left| \frac{\partial u}{\partial y}(y,t) \right|^2 dy \right) u_{xx}, \tag{1.3}$$

where u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. It is also related to the Carrier equation. In [2], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0} \int_0^L u^2(y,t) dy \right) u_{xx} = 0, \tag{1.4}$$

where $u(x,t)$ is the $x-$derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross - section of a string, L is the length of a string and and ρ is the density of a material.

The Kirchhoff–Carrier equations of the form (1.1), with various boundary conditions, have been extensively studied by many authors. We refer the reader to, e.g., [3], [5], [7] - [19], for many interesting results and further references.

In [3], the authors studied the existence of global solutions and exponential decay for a Kirchhoff -Carrier model with viscosity. In [16], the author investigated on the global existence, decay properties, and blow-up of solutions to the initial boundary value problem for the nonlinear Kirchhoff type. In [19], the viscoelastic equation of Kirchhoff type was considered and the authors etablished a new blow-up result for arbitrary positive initial energy, by using simple analysis techniques. In [7], Long used a linear approximation scheme to prove the existence and uniqueness of a local solution for a Kirchhoff-Carrier type equation of the form

$$\begin{cases} u_{tt} - B(t, \|u\|^2, \|u_x\|^2) u_{xx} = f(x,t,u,u_x,u_t,\|u\|^2, \|u_x\|^2), \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x). \end{cases} \tag{1.5}$$

In addition, an asymptotic expansion of the weak solution according to a small parameter was also considered. Results in [7] were later expanded in [9], and [17].

Triet et al. [18] investigated on the existence, decay properties, and blow-up of solutions of the Love-Kirchhoff wave equation associated with the Dirichlet boundary conditions. Recently, Ngoc et al. [14] demonstrated the existence and uniqueness of the solution of the boundary problem for a Kirchhoff-Carrier equation containing nonlinear Balakrishnan-Taylor terms as follows

$$\begin{cases} u_{tt} - \lambda u_{xxt} - \mu \left(t, \langle u_x(t), u_{xt}(t) \rangle, \|u(t)\|^2, \|u_x(t)\|^2 \right) u_{xx} \\ \quad = f\left(x,t,u,u_t,u_x, \langle u_x(t), u_{xt}(t) \rangle, \|u(t)\|^2, \|u_x(t)\|^2 \right),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.6}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, $\lambda > 0$ is given constant. In case $\mu = B(\|u_x(t)\|^2) + \sigma(\langle u_x(t), u_{xt}(t) \rangle)$ and $f \equiv -\lambda_1 u_t + f(u) + F(x,t)$, the authors have established sufficient conditions on the Balakrishnan-Taylor term $\sigma(\langle u_x(t), u_{xt}(t) \rangle)$ to obtain the exponential decay of the solution as $t \to \infty$.

Motivated by the above mentioned works, because of mathematical context, we study the unique existence and the exponential decay of the solution for Prob. (1.1). Our paper is organized as follows.

We present preliminaries in Section 2 and establish the unique existence of a weak solution in Section 3. In Section 4, Prob. (1.1) is considered with

$$\begin{aligned} \mu_1\left(x,t,u\left(x,t\right),\|u(t)\|^2,\|u_x(t)\|^2\right) &= \mu_1\left(\|u_x(t)\|^2\right), \\ \mu_2\left(t,s,\|u(s)\|^2,\|u_x(s)\|^2\right) &= \mu_2\left(t\right)g(t-s), \\ f\left(x,t,u,u_x,u_t\right) &= -\lambda_1 u_t - \sum_{i=1}^{N} K_i\left|u\right|^{p_i-2}u + F(x,t) \end{aligned} \tag{1.7}$$

and $\lambda_1 > 0$, $K_i > 0$, $p_i > 2$ $(i = 1, \cdots, N)$. If some auxiliary conditions are satisfied, then the solution decays exponentially as $t \to \infty$.

The obtained results have relatively generalized the results in [7], [8], [12], [17].

2 Preliminaries

Set $\Omega = (0,1)$, and $Q_T = \Omega \times (0,T)$, $T > 0$. Let us denote the standard function spaces by $C^m = C^m\left(\overline{\Omega}\right)$, $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$, $W^{m,p} = W^{m,p}(\Omega)$ (see [1]).

We define the scalar product $\langle\cdot,\cdot\rangle$ in $L^2 \equiv L^2(\Omega)$ by

$$\langle u,v\rangle = \int_0^1 u\left(x\right)v\left(x\right)dx,\ u,v \in L^2, \tag{2.1}$$

and the corresponding norm

$$\|u\| = \sqrt{\langle u,u\rangle}. \tag{2.2}$$

On H^1, we use the following norm

$$\|u\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.3}$$

Then, we have the following lemma.

Lemma 2.1. *(Lions [6])The imbeddings $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ and $H_0^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ are compact and*

$$\begin{aligned} (i)\ \ \|v\|_{C^0(\overline{\Omega})} &\le \sqrt{2}\,\|v\|_{H^1}, \quad \forall v \in H^1, \\ (ii)\ \ \|v\|_{C^0(\overline{\Omega})} &\le \|v_x\|, \quad \forall v \in H_0^1. \end{aligned} \tag{2.4}$$

Remark 2.2. *By (2.3) and (2.4), it is easy to prove that, on H_0^1, the two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent.*

We denote $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\int_0^T ||u(t)||_X^p\,dt < +\infty,\ 1 \le p < \infty,$$

and

$$\exists M > 0 : ||u(t)||_X \le M \text{ a.e } t \in (0,T),\ p = \infty,$$

with

$$||u||_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p \, dt\right)^{1/p}, & 1 \le p < \infty, \\ \underset{0<t<T}{ess\sup} ||u(t)||_X, & p = \infty, \end{cases}$$

where

$$\underset{0<t<T}{ess\sup} ||u(t)||_X = \inf\{M > 0 : ||u(t)||_X \le M \text{ a.e } t \in (0,T)\}.$$

Throughout this paper, we write

$$u(t),\ u_t(t) = \dot{u}(t),\ u_{tt}(t) = \ddot{u}(t),\ u_x(t) = \nabla u(t),\ u_{xx}(t) = \Delta u(t),$$

to denote

$$u(x,t),\ \frac{\partial u}{\partial t}(x,t),\ \frac{\partial^2 u}{\partial t^2}(x,t),\ \frac{\partial u}{\partial x}(x,t),\ \frac{\partial^2 u}{\partial x^2}(x,t),$$

respectively.

3 Main results

3.1 Local existence and uniqueness of a weak solution

The weak fomulation of Prob. (1.1) can be given in the following manner: Find $u \in \hat{W}_T = \{u \in L^\infty(0,T;H^2 \cap H_0^1) : u' \in L^\infty(0,T;H^2 \cap H_0^1) \text{ and } u'' \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)\}$, such that u satisfies the following variational equation

$$\langle u''(t), v\rangle + \lambda \langle u_x'(t), v_x\rangle + \langle \mu_1[u](t)u_x(t), v_x\rangle = \int_0^t \mu_2[u](t,s)\langle u_x(s), v_x\rangle \, ds + \langle f[u](t), v\rangle, \quad (3.1)$$

for all $v \in H_0^1$, and a.e. $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \quad (3.2)$$

where

$$\begin{aligned} f[u](x,t) &= f(x,t,u(x,t),u_x(x,t),u_t(x,t)), \\ \mu_1[u](x,t) &= \mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right), \\ \mu_2[u](t,s) &= \mu_2\left(t,s,\|u(s)\|^2,\|u_x(s)\|^2\right). \end{aligned} \quad (3.3)$$

Remark 3.1. *The weak solutions have following properties*

$$\begin{aligned} u &\in L^\infty(0,T;H^2 \cap H_0^1) \cap C^0([0,T];H^2 \cap H_0^1) \cap C^1([0,T];H_0^1), \\ u' &\in L^\infty(0,T;H^2 \cap H_0^1) \cap C^0([0,T];H_0^1), \\ u'' &\in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1). \end{aligned}$$

Now, let $T^* > 0$ be fixed and we make the following assumptions

(H_1) $\tilde{u}_0,\ \tilde{u}_1 \in H^2 \cap H_0^1$;

(H_2) $\mu_1 \in C^2([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$,
$\mu_1(x,t,y,z_1,z_2) \ge \mu_{1*} > 0,\ \forall (x,t,y,z_1,z_2) \in [0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$;

(H_3) $\mu_2 \in C^1(\Delta_{T^*} \times \mathbb{R}_+^2),\ \Delta_{T^*} = \{(t,s) : 0 \le s \le t \le T^*\}$;

(H_4) $f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^3)$ *such that*
$$f(0,t,0,y,0) = f(1,t,0,y,0) = 0,\ \forall (t,y) \in [0,T^*] \times \mathbb{R}.$$

For each $T \in (0, T^*]$, we put

$$W_T = \{v \in L^\infty(0,T;H^2 \cap H_0^1) : v' \in L^\infty(0,T;H^2 \cap H_0^1),\ v'' \in L^2(0,T;H_0^1)\}, \tag{3.4}$$
$$W_1(T) = \{v \in L^\infty(0,T;H_0^1) : v' \in L^2(0,T;H_0^1) \cap L^\infty(0,T;L^2)\}.$$

It is well known that $W_T,\ W_1(T)$ are Banach spaces with respect to the norms (Lions [6])

$$\|v\|_{W_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)};\ \ \|v'\|_{L^\infty(0,T;H^2\cap H_0^1)};\ \ \|v''\|_{L^2(0,T;H_0^1)}\}, \tag{3.5}$$
$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|v'\|_{L^2(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}.$$

For each $M > 0$, we denote

$$W(M,T) = \{v \in W_T : \|v\|_{W_T} \le M\}, \tag{3.6}$$
$$W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}.$$

We next establish the linear recurrent sequence $\{u_m\}$ as follows:
We shall choose the first term $u_0 \equiv 0$ and suppose that

$$u_{m-1} \in W_1(M,T). \tag{3.7}$$

We find $u_m \in W_1(M,T)$ is a solution of the variational problem associating Prob. (1.1) as follows

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda \langle u_{mx}'(t), v_x\rangle + \langle \mu_{1m}(t) u_{mx}(t), v_x\rangle = \displaystyle\int_0^t \mu_{2m}(t,s) \langle u_{mx}(s), v_x\rangle\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \langle F_m(t), v\rangle\,,\ \ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.8}$$

where

$$\mu_{1m}(x,t) = \mu_1[u_{m-1}](x,t) = \mu_1(x,t,u_{m-1}(x,t)\,, \|u_{m-1}(t)\|^2\,, \|\nabla u_{m-1}(t)\|^2), \tag{3.9}$$
$$\mu_{2m}(t,s) = \mu_2[u_{m-1}](t,s) = \mu_2(t,s,\|u_{m-1}(s)\|^2\,, \|\nabla u_{m-1}(s)\|^2),$$
$$F_m(x,t) = f[u_{m-1}](x,t) = f\left(x,t,u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t)\right).$$

Then, we have the following theorem concerning the existence and uniqueness of a weak solution.

Theorem 3.2. *Let $(H_1) - (H_4)$ hold. Then, there exist positive constants M, T such that*
(i) *Prob. (1.1) has a unique weak solution $u \in W_1(M,T)$.*
(ii) *The linear recurrent sequence $\{u_m\}$ defined by (3.7)-(3.9) converges to the solution u strongly in the space $W_1(T)$ with the estimate*

$$\|u_m - u\|_{W_1(T)} \le \frac{M}{1-k_T} k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.10}$$

where $k_T \in [0,1)$ is a constant independent of m.

Proof. The proof is similar to the argument in [14], so we omit the details. □

3.2 Exponential decay of the solution

In this section, Prob. (1.1) is considered with $\mu_1 = \mu_1\left(\|u_x(t)\|^2\right)$, $\mu_2 = \mu_2\,(t)\,g(t-s)$,

$$f\,(x,t,u,u_x,u_t) = -\lambda_1 u_t - \sum_{i=1}^{N} K_i\,|u|^{p_i-2}\,u + F(x,t),$$

it means that Prob. (1.1) becomes

$$\begin{cases} u_{tt} - \lambda u_{txx} - \mu_1\left(\|u_x(t)\|^2\right)u_{xx} + \mu_2\,(t)\displaystyle\int_0^t g(t-s)u_{xx}(x,s)ds \\ \qquad +\lambda_1 u_t + \displaystyle\sum_{i=1}^{N} K_i\,|u|^{p_i-2}\,u = F\,(x,t)\,,\ 0<x<1,\ t>0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{3.11}$$

where $p_i > 2$, $K_i > 0$, $\lambda > 0$, $\lambda_1 > 0$ are given constants, and F, μ_1, μ_2, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions specified later. Here, we shall prove the exponential decay of the weak solution of Prob. (3.11) as $t \to \infty$, i.e., there exist positive constants $\bar{C}$, $\bar{\gamma}$, such that

$$\|u'\,(t)\|^2 + \|u_x\,(t)\|^2 + \sum_{i=1}^{N}\|u(t)\|_{L^{p_i}}^{p_i} \le \bar{C}e^{-\bar{\gamma}t},\ \text{for all } t \ge 0. \tag{3.12}$$

First, for $T > 0$, we suppose that

- (H_1) $\tilde{u}_0,\ \tilde{u}_1 \in H^2 \cap H_0^1$;
- $(\hat{H}_2)$ $\mu_1 \in C^1(\mathbb{R}_+)$ and there exist two positive constants μ_{1*}, $\bar{\mu}_1^*$ such that
  - (i) $\mu_1(z) \ge \mu_{1*} > 0,\ \forall z \ge 0$;
  - (ii) $|z\mu_1'\,(z)| \le \bar{\mu}_1^*\mu_1\,(z)\,,\ \forall z \ge 0$,
- $(\hat{H}_3)$ $\mu_2 \in C^1([0,T])$;
- $(\hat{H}_4)$ $g \in C^1([0,T])$;
- $(\hat{H}_5)$ $F \in C^1(\bar{Q}_T)$.

Then, we have the following theorem.

Theorem 3.3. *Let $p_i > 2$, $K_i > 0$ $(i = 1,\cdots,N)$, $\lambda > 0$, $\lambda_1 > 0$. Suppose that (H_1), $(\hat{H}_2)$–$(\hat{H}_5)$ hold. Then, Prob. (3.11) has a unique solution u such that*

$$u \in L^\infty(0,T;H^2\cap H_0^1),\ u' \in L^\infty(0,T;H^2\cap H_0^1) \text{ and } u'' \in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1). \tag{3.13}$$

Note that from (3.13), we deduce

$$\begin{cases} u \in C^0([0,T];H^2\cap H_0^1)\cap C^1([0,T];H_0^1)\cap L^\infty(0,T;H^2\cap H_0^1), \\ u' \in C^0([0,T];H_0^1)\cap L^\infty(0,T;H^2\cap H_0^1), \\ u'' \in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1). \end{cases} \tag{3.14}$$

If $\tilde{u}_0$, $\tilde{u}_1$, g and F are less smooth as follows

- $(\bar{H}_1)$ $(\tilde{u}_0,\tilde{u}_1) \in H_0^1 \times L^2$;
- $(\bar{H}_4)$ $g \in L^2(0,T)$;
- $(\bar{H}_5)$ $F \in L^2(Q_T)$,

then we also obtain the existence of a weak solution (as in Theorem 3.4 below) $u \in \tilde{W}_1(T) = \{u \in C^0([0,T];H_0^1) \cap C^1([0,T];L^2) : u' \in L^2(0,T;H_0^1)\}$.

We note more that $\tilde{W}_1(T)$ is a Banach space (Lions [6]) which respects to the norm

$$\|v\|_{\tilde{W}_1(T)} = \|v\|_{C^0\left([0,T];H_0^1\right)} + \|v\|_{C^1([0,T];L^2)} + \|v'\|_{L^2(0,T;H_0^1)}. \tag{3.15}$$

Theorem 3.4. *Let $p_i > 2$, $K_i > 0$ $(i = 1, \cdots, N)$, $\lambda > 0$, $\lambda_1 > 0$ and the assumptions $\left(\bar{H}_1\right)$, $(\hat{H}_2)$, $(\hat{H}_3)$, $\left(\bar{H}_4\right)$, $(\bar{H}_5)$ hold. Then, Prob. (3.11) has a unique weak solution u in $\tilde{W}_1(T)$.*

Proof. Consider $(\tilde{u}_0, \tilde{u}_1, g, F) \in H_0^1 \times L^2 \times L^2(0,T) \times L^2(Q_T)$. By using standard arguments of density, there exists a sequence $\{(\tilde{u}_{0m}, \tilde{u}_{1m}, g_m, F_m)\} \subset C_c^\infty\left(\bar{\Omega}\right) \times C_c^\infty(\Omega) \times C_c^\infty(0,T) \times C_c^\infty(Q_T)$, such that

$$\begin{cases} (\tilde{u}_{0m}, \tilde{u}_{1m}) \to (\tilde{u}_0, \tilde{u}_1) & \text{strongly in} \quad H_0^1 \times L^2, \\ g_m \to g & \text{strongly in} \quad L^2(0,T), \\ F_m \to F & \text{strongly in} \quad L^2(Q_T). \end{cases} \tag{3.16}$$

Then, for each $m \in \mathbb{N}$, there exists a unique function u_m under the conditions of Theorem3.3. Thus, we can verify

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda \langle u_{mx}'(t), v_x\rangle + \mu_1\left(\|u_{mx}(t)\|^2\right)\langle u_{mx}(t), v_x\rangle \\ \qquad +\lambda_1 \langle u_m'(t), v\rangle + \sum_{i=1}^N K_i \left\langle |u_m(t)|^{p_i-2} u_m(t), v\right\rangle \\ \quad = \mu_2(t)\int_0^t g_m(t-s)\langle u_{mx}(s), v_x\rangle ds + \langle F_m(t), v\rangle, \ \forall v \in H_0^1, \text{ a.e., } t \in (0,T_m), \\ u_m(0) = \tilde{u}_{0m}, \ u_m'(0) = \tilde{u}_{1m}, \end{cases} \tag{3.17}$$

and

$$\begin{cases} u_m \in C^0([0,T_m];H^2 \cap H_0^1) \cap C^1([0,T_m];H_0^1) \cap L^\infty(0,T_m;H^2 \cap H_0^1), \\ u_m' \in C^0([0,T_m];H_0^1) \cap L^\infty(0,T_m;H^2 \cap H_0^1), \\ u_m'' \in L^\infty(0,T_m;L^2) \cap L^2(0,T_m;H_0^1). \end{cases} \tag{3.18}$$

A priori estimates.

We take $v = u_m'(t)$ in (3.17), we have

$$\begin{aligned} S_m(t) &= S_m(0) + 2\int_0^t \langle F_m(s), u_m'(s)\rangle ds \\ &\quad + 2\int_0^t \mu_2(\tau) d\tau \int_0^\tau g_m(\tau - s)\langle u_{mx}(s), u_{mx}'(\tau)\rangle ds \\ &= S_m(0) + I_1 + I_2, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} S_m(t) &= \|u_m'(t)\|^2 + 2\lambda\int_0^t \|u_{mx}'(s)\|^2 ds \\ &\quad + 2\lambda_1 \int_0^t \|u_m'(s)\|^2 ds + \int_0^{\|u_{mx}(t)\|^2} \mu_1(z)dz \\ &\quad + \sum_{i=1}^N \frac{2K_i}{p_i}\|u_m(t)\|_{L^{p_i}}^{p_i}. \end{aligned} \tag{3.20}$$

Noting that, from

$$S_m(t) \geq \gamma_* \left(\|u'_m(t)\|^2 + \|u_{mx}(t)\|^2 + \int_0^t \|u'_{mx}(s)\|^2 ds + \int_0^t \|u'_m(s)\|^2 ds \right), \tag{3.21}$$

with $\gamma_* = \min\{1, \mu_{1*}, 2\lambda, 2\lambda_1\} > 0$, we will estimate the integrals I_1, I_2 in the right hand side of (3.19) as follows:

$$\begin{aligned} I_1 &= 2\int_0^t \langle F_m(s), u'_m(s)\rangle ds \leq \|F_m\|^2_{L^2(Q_T)} + \int_0^t S_m(s) ds; \qquad (3.22) \\ I_2 &= 2\int_0^t \mu_2(\tau) d\tau \int_0^\tau g_m(\tau - s) \langle u_{mx}(s), u'_{mx}(\tau)\rangle ds \\ &\leq 2\|\mu_2\|_{L^\infty} \int_0^t \|u'_{mx}(\tau)\| d\tau \int_0^\tau |g_m(\tau - s)| \|u_{mx}(s)\| ds \\ &\leq \frac{2}{\gamma_*} \|\mu_2\|_{L^\infty} \|g_m\|_{L^2} \sqrt{T^*} \sqrt{S_m(t)} \left[\int_0^t S_m(s) ds\right]^{1/2} \\ &\leq \frac{1}{2} S_m(t) + \frac{2}{\gamma_*^2} \|\mu_2\|^2_{L^\infty} \|g_m\|^2_{L^2} T^* \int_0^t S_m(s) ds, \end{aligned}$$

where $\|\mu_2\|_{L^\infty} = \|\mu_2\|_{L^\infty(0,T)}$, $\|g_m\|_{L^2} = \|g_m\|_{L^2(0,T)}$.

From (3.22) and (3.19), we deduce that

$$\begin{aligned} S_m(t) &\leq 2S_m(0) + 2\|F_m\|^2_{L^2(Q_T)} \\ &+ 2\left(1 + \frac{2T^*}{\gamma_*^2} \|\mu_2\|^2_{L^\infty} \|g_m\|^2_{L^2}\right) \int_0^t S_m(s) ds, \ 0 \leq t \leq T_m \leq T. \end{aligned} \tag{3.23}$$

Since

$$S_m(0) = \|\tilde{u}_{1m}\|^2 + \int_0^{\|\tilde{u}_{0mx}\|^2} \mu_1(z) dz + \sum_{i=1}^N \frac{2K_i}{p_i} \|\tilde{u}_{0m}\|^{p_i}_{L^{p_i}},$$

and from (3.16), there exists a posive constant $\tilde{C}_1$ which is independent of m such that

$$2S_m(0) + 2\|F_m\|^2_{L^2(Q_T)} + 2\left(1 + \frac{2T^*}{\gamma_*^2} \|\mu_2\|^2_{L^\infty} \|g_m\|^2_{L^2}\right) \leq \tilde{C}_1, \ \forall m \in \mathbb{N}. \tag{3.24}$$

From (3.23) and (3.24), it implies that

$$S_m(t) \leq \tilde{C}_1 + \tilde{C}_1 \int_0^t S_m(s) ds, \ 0 \leq t \leq T_m \leq T, \ \forall m \in \mathbb{N}. \tag{3.25}$$

By Gronwall's lemma, it follows from (3.25) that

$$S_m(t) \leq \tilde{C}_1 \exp\left(T\tilde{C}_1\right) = \tilde{C}_T, \ \forall t \in [0, T]. \tag{3.26}$$

Therefore, we can take $T_m = T$ for all $m \in \mathbb{N}$. Now, we shall prove that $\{u_m\}$ is a Cauchy sequence in $C^0([0,T]; H_0^1) \cap C^1([0,T]; L^2)$. First, we put

$$\begin{cases} w_{m,l} = u_m - u_l, \ F_{m,l} = F_m - F_l, \ g_{m,l} = g_m - g_l, \\ \tilde{u}_{0m,l} = \tilde{u}_{0m} - \tilde{u}_{0l}, \ \tilde{u}_{1m,l} = \tilde{u}_{1m} - \tilde{u}_{1l}, \end{cases} \tag{3.27}$$

and from (3.17), we deduce that

$$\begin{cases} \left\langle w''_{m,l}(t), v\right\rangle + \lambda \left\langle w'_{m,lx}(t), v_x\right\rangle + \mu_1\left(\|u_{mx}(t)\|^2\right)\left\langle \nabla W_{m,l}(t), v_x\right\rangle \\ \qquad + \left[\mu_1\left(\|u_{mx}(t)\|^2\right) - \mu_1\left(\|u_{lx}(t)\|^2\right)\right]\left\langle u_{lx}(t), v_x\right\rangle \\ \qquad + \lambda_1\left\langle w'_{m,l}(t), v\right\rangle + \sum_{i=1}^{N} K_i\left\langle |u_m(t)|^{p_i-2}u_m(t) - |u_l(t)|^{p_i-2}u_l(t), v\right\rangle \\ = \mu_2(t)\int_0^t g_m(t-s)\left\langle \nabla w_{m,l}(s), v_x\right\rangle ds + \mu_2(t)\int_0^t g_{m,l}(t-s)\left\langle u_{lx}(s), v_x\right\rangle ds \\ \qquad + \left\langle F_{m,l}(t), v\right\rangle, \ \forall v \in H_0^1, \text{ a.e., } t \in (0,T), \\ w_{m,l}(0) = \tilde{u}_{0m,l}, \ w'_{m,l}(0) = \tilde{u}_{1m,l}. \end{cases} \tag{3.28}$$

Setting $v = w'_{m,l} = u'_m - u'_l$ in (3.28) and then integrating with respect to the time variable from 0 to t, we have

$$\begin{aligned} S_{m,l}(t) &= S_{m,l}(0) + 2\int_0^t \left\langle F_{m,l}(s), w'_{m,l}(s)\right\rangle ds \\ &\quad - 2\sum_{i=1}^{N} K_i \int_0^t \left\langle |u_m(s)|^{p_i-2}u_m(s) - |u_l(s)|^{p_i-2}u_l(s), w'_{m,l}(s)\right\rangle ds \\ &\quad + 2\int_0^t \mu'_1\left(\|u_{mx}(s)\|^2\right)\left\langle u_{mx}(s), u'_{mx}(s)\right\rangle \|\nabla w_{m,l}(s)\|^2 ds \\ &\quad - 2\int_0^t \left[\mu_1\left(\|u_{mx}(s)\|^2\right) - \mu_1\left(\|u_{lx}(s)\|^2\right)\right]\left\langle u_{lx}(s), \nabla w'_{m,l}(s)\right\rangle ds \\ &\quad + 2\int_0^t \mu_2(\tau)\, d\tau \int_0^\tau g_m(\tau - s)\left\langle \nabla w_{m,l}(s), \nabla w'_{m,l}(\tau)\right\rangle ds \\ &\quad + 2\int_0^t \mu_2(\tau)\, d\tau \int_0^\tau g_{m,l}(\tau - s)\left\langle u_{lx}(s), \nabla w'_{m,l}(\tau)\right\rangle ds \\ &\equiv S_{m,l}(0) + J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} S_{m,l}(t) &= \left\|w'_{m,l}(t)\right\|^2 + \mu_1\left(\|u_{mx}(t)\|^2\right)\|\nabla w_{m,l}(t)\|^2 \\ &\quad + 2\lambda\int_0^t \left\|\nabla w'_{m,l}(s)\right\|^2 ds + 2\lambda_1\int_0^t \left\|w'_{m,l}(s)\right\|^2 ds, \\ S_{m,l}(0) &= \left\|w'_{m,l}(0)\right\|^2 + \mu_1\left(\|\tilde{u}_{0mx}\|^2\right)\|\nabla w_{m,l}(0)\|^2 \\ &= \|\tilde{u}_{1m} - \tilde{u}_{1l}\|^2 + \mu_1\left(\|\tilde{u}_{0mx}\|^2\right)\|\tilde{u}_{0mx} - \tilde{u}_{0lx}\|^2 \to 0 \text{ as } m,\ l \to \infty. \end{aligned} \tag{3.30}$$

Note that

$$S_{m,l}(t) \geq \gamma_*\left(\left\|w'_{m,l}(t)\right\|^2 + \|\nabla w_{m,l}(t)\|^2 + \int_0^t \left\|\nabla w'_{m,l}(s)\right\|^2 ds + \int_0^t \left\|w'_{m,l}(s)\right\|^2 ds\right),$$

where $\gamma_* = \min\{1, \mu_{1*}, 2\lambda, 2\lambda_1\} > 0$, the terms $J_1 - J_6$ on the right-hand side of (3.29) are estimated as follows.

Estimate of J_1. We have

$$J_1 = 2\int_0^t \left\langle F_{m,l}(s), w'_{m,l}(s)\right\rangle ds \le \|F_{m,l}\|^2_{L^2(Q_{T_*})} + \int_0^t S_{m,l}(s)\, ds. \tag{3.31}$$

Estimate of J_2. From (3.26) and using the following inequality

$$\left||x|^{p-2}x - |y|^{p-2}y\right| \le (p-1)M_1^{p-2}|x-y|, \ \forall x, y \in [-M_1, M_1], \ \forall M_1 > 0, \ \forall p > 2,$$

with $M_1 = \sqrt{\frac{\tilde{C}_T}{\gamma_*}}$, we have

$$\begin{aligned} J_2 &= -2\sum_{i=1}^N K_i \int_0^t \left\langle |u_m(s)|^{p_i-2} u_m(s) - |u_l(s)|^{p_i-2} u_l(s), w'_{m,l}(s)\right\rangle ds \\ &\le 2\sum_{i=1}^N K_i \int_0^t \left\||u_m(s)|^{p_i-2} u_m(s) - |u_l(s)|^{p_i-2} u_l(s)\right\| \left\|w'_{m,l}(s)\right\| ds \\ &\le 2\sum_{i=1}^N K_i (p_i-1)\left(\sqrt{\frac{\tilde{C}_T}{\gamma_*}}\right)^{p_i-2} \int_0^t \|w_{m,l}(s)\| \left\|w'_{m,l}(s)\right\| ds \\ &\le \sum_{i=1}^N K_i \frac{p_i-1}{\gamma_*}\left(\sqrt{\frac{\tilde{C}_T}{\gamma_*}}\right)^{p_i-2} \int_0^t S_{m,l}(s)\, ds. \end{aligned} \tag{3.32}$$

Estimate of J_3. Setting $\mu_1^{**} = \sup\{|\mu_1'(z)| : 0 \le z \le \frac{\tilde{C}_T}{\gamma_*}\}$, from $(\hat{H}_2(ii))$ and the following inequalities

$$\begin{aligned} S_{m,l}(t) &= \left\|w'_{m,l}(t)\right\|^2 + \mu_1\left(\|u_{mx}(t)\|^2\right)\|\nabla w_{m,l}(t)\|^2 \\ &\quad + 2\lambda\int_0^t \left\|\nabla w'_{m,l}(s)\right\|^2 ds + 2\lambda_1 \int_0^t \left\|w'_{m,l}(s)\right\|^2 ds \\ &\ge \mu_1\left(\|u_{mx}(t)\|^2\right)\|\nabla w_{m,l}(t)\|^2 \\ &= \sqrt{\mu_1\left(\|u_{mx}(t)\|^2\right)}\sqrt{\mu_1\left(\|u_{mx}(t)\|^2\right)}\|\nabla w_{m,l}(t)\|^2 \\ &\ge \sqrt{\mu_{1*}}\sqrt{\mu_1\left(\|u_{mx}(t)\|^2\right)}\|\nabla w_{m,l}(t)\|^2, \end{aligned}$$

and

$$\begin{aligned} \left|\mu_1'\left(\|u_{mx}(s)\|^2\right)\right| \|u_{mx}(s)\| &\le \sqrt{\left|\mu_1'\left(\|u_{mx}(s)\|^2\right)\right|}\sqrt{\left|\mu_1'\left(\|u_{mx}(s)\|^2\right)\right| \|u_{mx}(s)\|^2} \\ &\le \sqrt{\mu_1^{**}}\sqrt{\bar{\mu}_1^*\mu_1\left(\|u_{mx}(s)\|^2\right)} \\ &= \sqrt{\mu_1^{**}\bar{\mu}_1^*}\sqrt{\mu_1\left(\|u_{mx}(s)\|^2\right)}, \end{aligned}$$

we have the estimate of J_3 as follows

$$\begin{aligned} J_3 &= 2\int_0^t \mu_1'\left(\|u_{mx}(s)\|^2\right)\langle u_{mx}(s), u_{mx}'(s)\rangle \|\nabla w_{m,l}(s)\|^2 ds \\ &\le 2\int_0^t \left|\mu_1'\left(\|u_{mx}(s)\|^2\right)\right| \|u_{mx}(s)\| \|u_{mx}'(s)\| \|\nabla w_{m,l}(s)\|^2 ds \\ &= 2\int_0^t \|u_{mx}'(s)\| \left|\mu_1'\left(\|u_{mx}(s)\|^2\right)\right| \|u_{mx}(s)\| \|\nabla w_{m,l}(s)\|^2 ds \\ &\le 2\sqrt{\mu_1^{**}\bar{\mu}_1^*}\int_0^t \|u_{mx}'(s)\| \sqrt{\mu_1\left(\|u_{mx}(s)\|^2\right)} \|\nabla w_{m,l}(s)\|^2 ds \\ &\le 2\sqrt{\mu_1^{**}\bar{\mu}_1^*}\int_0^t \|u_{mx}'(s)\| \frac{S_{m,l}(s)}{\sqrt{\mu_{1*}}} ds \\ &= 2\sqrt{\frac{\mu_1^{**}\bar{\mu}_1^*}{\mu_{1*}}}\int_0^t \|u_{mx}'(s)\| S_{m,l}(s)\, ds. \end{aligned} \tag{3.33}$$

Estimate of J_4. Using the mean value theorem of Lagrange, there exists a positive constant $\theta \in (0,1)$ such that

$$\begin{aligned} &\mu_1\left(\|u_{mx}(s)\|^2\right) - \mu_1\left(\|u_{lx}(s)\|^2\right) \\ &= \left[\|u_{mx}(s)\|^2 - \|u_{lx}(s)\|^2\right]\mu_1'\left(\theta\|u_{mx}(s)\|^2 + (1-\theta)\|u_{lx}(s)\|^2\right). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} \left|\mu_1\left(\|u_{mx}(s)\|^2\right) - \mu_1\left(\|u_{lx}(s)\|^2\right)\right| &\le \mu_1^{**}\left|\|u_{mx}(s)\|^2 - \|u_{lx}(s)\|^2\right| \\ &\le 2\sqrt{\frac{\tilde{C}_T}{\gamma_*}}\mu_1^{**}\|\nabla w_{m,l}(s)\|. \end{aligned}$$

It leads to

$$\begin{aligned} |J_4| &= \left|-2\int_0^t \left[\mu_1\left(\|u_{mx}(s)\|^2\right) - \mu_1\left(\|u_{lx}(s)\|^2\right)\right]\left\langle u_{lx}(s), \nabla w_{m,l}'(s)\right\rangle ds\right| \\ &\le 2\int_0^t \left|\mu_1\left(\|u_{mx}(s)\|^2\right) - \mu_1\left(\|u_{lx}(s)\|^2\right)\right| \|u_{lx}(s)\| \left\|\nabla w_{m,l}'(s)\right\| ds \\ &\le 4\frac{\tilde{C}_T}{\gamma_*}\mu_1^{**}\int_0^t \|\nabla w_{m,l}(s)\| \left\|\nabla w_{m,l}'(s)\right\| ds \\ &\le \beta\gamma_*\int_0^t \left\|\nabla w_{m,l}'(s)\right\|^2 ds + 4\frac{\tilde{C}_T^2}{\beta\gamma_*^3}\left(\mu_1^{**}\right)^2\int_0^t \|\nabla w_{m,l}(s)\|^2 ds \\ &\le \beta S_{m,l}(t) + 4\frac{\tilde{C}_T^2}{\beta\gamma_*^4}\left(\mu_1^{**}\right)^2\int_0^t S_{m,l}(s)\, ds. \end{aligned} \tag{3.34}$$

Estimate of J_5.

$$\begin{aligned} J_5 &= 2\int_0^t \mu_2(\tau)\,d\tau \int_0^\tau g_m(\tau-s)\left\langle \nabla w_{m,l}(s), \nabla w'_{m,l}(\tau)\right\rangle ds \\ &\le 2\|\mu_2\|_{L^\infty}\int_0^t \left\|\nabla w'_{m,l}(\tau)\right\| d\tau \int_0^\tau |g_m(\tau-s)|\,\|\nabla w_{m,l}(s)\|\,ds \\ &\le 2\|\mu_2\|_{L^\infty}\sqrt{T}\,\|g_m\|_{L^2}\left[\int_0^t \left\|\nabla w'_{m,l}(\tau)\right\|^2 d\tau\right]^{1/2}\left[\int_0^t \|\nabla w_{m,l}(s)\|^2 ds\right]^{1/2} \\ &\le \frac{2}{\gamma_*}\|\mu_2\|_{L^\infty}\sqrt{T}\,\|g_m\|_{L^2}\left[S_{m,l}(t)\right]^{1/2}\left[\int_0^t S_{m,l}(s)\,ds\right]^{1/2} \\ &\le \beta S_{m,l}(t) + \frac{1}{\beta\gamma_*}\|\mu_2\|^2_{L^\infty}\,T\,\|g_m\|^2_{L^2}\int_0^t S_{m,l}(s)\,ds, \end{aligned} \tag{3.35}$$

with $\|\mu_2\|_{L^\infty} = \|\mu_2\|_{L^\infty(0,T)}$, $\|g_m\|_{L^2} = \|g_m\|_{L^2(0,T)}$.

Estimate of J_6.

$$\begin{aligned} J_6 &= 2\int_0^t \mu_2(\tau)\,d\tau \int_0^\tau g_{m,l}(\tau-s)\left\langle u_{lx}(s), \nabla w'_{m,l}(\tau)\right\rangle ds \\ &\le 2\|\mu_2\|_{L^\infty}\sqrt{T}\,\|g_{m,l}\|_{L^2}\left[\int_0^t \left\|\nabla w'_{m,l}(\tau)\right\|^2 d\tau\right]^{1/2}\left[\int_0^t \|u_{lx}(s)\|^2 ds\right]^{1/2} \\ &\le 2\|\mu_2\|_{L^\infty}\sqrt{T}\,\|g_{m,l}\|_{L^2}\left[\frac{S_{m,l}(t)}{\gamma_*}\right]^{1/2}\sqrt{\frac{\tilde{C}_T}{\gamma_*}} \\ &= \frac{2}{\gamma_*}\sqrt{\tilde{C}_T}\,\|\mu_2\|_{L^\infty}\sqrt{T}\,\|g_{m,l}\|_{L^2}\sqrt{S_{m,l}(t)} \le \beta S_{m,l}(t) + \frac{1}{\beta\gamma_*^2}\tilde{C}_T\|\mu_2\|^2_{L^\infty}\,T\,\|g_{m,l}\|^2_{L^2}. \end{aligned} \tag{3.36}$$

Choosing $\beta = \frac{1}{6}$, from (3.31)-(3.36), it follows from (3.29) that

$$S_{m,l}(t) \le R_T(m,l) + \int_0^t \bar{R}_m(s) S_{m,l}(s)\,ds, \tag{3.37}$$

where

$$\begin{aligned} R_T(m,l) &= 2S_{m,l}(0) + 2\|F_{m,l}\|^2_{L^2(Q_{T_*})} + \frac{12}{\gamma_*^2}\tilde{C}_T\|\mu_2\|^2_{L^\infty}\,T\,\|g_{m,l}\|^2_{L^2}, \\ \bar{R}_m(s) &= \tilde{D}_T^{(1)} + \tilde{D}_T^{(m)} + 4\sqrt{\frac{\mu_1^{**}\bar{\mu}_1^*}{\mu_{1*}}}\,\|u'_{mx}(s)\|, \\ \tilde{D}_T^{(1)} &= 2\left[1 + \sum_{i=1}^N K_i\frac{p_i-1}{\gamma_*}\left(\sqrt{\frac{\tilde{C}_T}{\gamma_*}}\right)^{p_i-2} + \frac{24\tilde{C}_T^2}{\gamma_*^4}(\mu_1^{**})^2\right], \\ \tilde{D}_T^{(m)} &= \frac{12}{\gamma_*}T\,\|\mu_2\|^2_{L^\infty}\|g_m\|^2_{L^2}. \end{aligned} \tag{3.38}$$

From (3.16), we get

$$\begin{aligned} R_T(m,l) &= 2S_{m,l}(0) + 2\|F_{m,l}\|^2_{L^2(Q_{T_*})} + \frac{12}{\gamma_*^2}T\tilde{C}_T\|\mu_2\|^2_{L^\infty}\|g_{m,l}\|^2_{L^2} \\ &\to 0, \text{ as } m,\ l \to +\infty; \end{aligned} \tag{3.39}$$

and there exists a constants $\tilde{D}_T^{(2)} > 0$ such that

$$\tilde{D}_T^{(m)} = \frac{12}{\gamma_*} \|\mu_2\|_{L^\infty}^2 T \|g_m\|_{L^2}^2 \leq \tilde{D}_T^{(2)}, \ \forall m \in \mathbb{N}.$$

From (3.38), we deduce that

$$\begin{aligned}\int_0^t \bar{R}_m(s)ds &\leq \int_0^t \left[\tilde{D}_T^{(1)} + \tilde{D}_T^{(m)} + 4\bar{\mu}_1^* \|u'_{mx}(s)\|\right] ds \\ &\leq T\left(\tilde{D}_T^{(1)} + \tilde{D}_T^{(m)}\right) + 4\bar{\mu}_1^* \sqrt{T} \left(\int_0^T \|u'_{mx}(s)\|^2 ds\right)^{1/2} \\ &\leq T\tilde{D}_T^{(2)} + 4\bar{\mu}_1^* \sqrt{T} \sqrt{\frac{\tilde{C}_T}{\gamma_*}} \equiv \tilde{D}_T^{(3)}, \ \forall m \in \mathbb{N}.\end{aligned}$$

Using Gronwall' lemma, and from (3.37), we obtain

$$\begin{aligned}S_{m,l}(t) &\leq R_T(m,l) \exp\left(\int_0^t \bar{R}_m(s)ds\right) \\ &\leq R_T(m,l) \exp\left(T\tilde{D}_T^{(3)}\right) \equiv \tilde{D}_T^{(4)} R_T(m,l), \ \forall m,l \in \mathbb{N}, \ \forall t \in [0,T].\end{aligned} \tag{3.40}$$

Combining (3.30), (3.39), (3.40), it implies that

$$\begin{aligned}&\sup_{0\leq t\leq T} \|u'_m(t) - u'_l(t)\|^2 + \sup_{0\leq t\leq T} \|\nabla u_m(t) - \nabla u_l(t)\|^2 + \|u'_m - u'_l\|^2_{L^2(0,T;H_0^1)} \\ &\leq \frac{3}{\gamma_*} \tilde{D}_T^{(4)} R_T(m,l) \to 0, \ \text{as } m, \ l \longrightarrow +\infty.\end{aligned} \tag{3.41}$$

We deduce that $\{u_m\}$ is a Cauchy in $\tilde{W}_1(T)$, there exists a function $u \in \tilde{W}_1(T)$ such that

$$u_m \to u \text{ strongly in } \tilde{W}_1(T). \tag{3.42}$$

From (3.42) and using the inequality $\left||x|^{p-2} x - |y|^{p-2} y\right| \leq (p-1)M_1^{p-2} |x-y|, \ \forall x, y \in [-M_1, M_1], \ \forall M_1 > 0, \forall p > 2$, with $M_1 = \sqrt{\frac{\tilde{C}_T}{\gamma_*}}$, we have

$$\left\||u_m|^{p_i-2} u_m - |u|^{p_i-2} u\right\|_{L^\infty(Q_T)} \leq (p_i - 1) \left(\sqrt{\frac{\tilde{C}_T}{\gamma_*}}\right)^{p_i-2} \sqrt{T} \sup_{0\leq t\leq T} \|u_{mx}(t) - u_x(t)\| \to 0, \tag{3.43}$$

so

$$|u_m|^{p_i-2} u_m \to |u|^{p_i-2} u \ \text{ strongly in } L^\infty(Q_T). \tag{3.44}$$

Therefore

$$\sum_{i=1}^N K_i |u_m|^{p_i-2} u_m \to \sum_{i=1}^N K_i |u|^{p_i-2} u \ \text{ strongly in } L^\infty(Q_T). \tag{3.45}$$

On the other hand, we have

$$\sup_{0\leq t\leq T} \left|\mu_1\left(\|u_{mx}(t)\|^2\right) - \mu_1\left(\|u_x(t)\|^2\right)\right| \leq 2\sqrt{\frac{\tilde{C}_T}{\gamma_*}} \mu_1^{**} \sup_{0\leq t\leq T} \|u_{mx}(t) - u_x(t)\| \to 0.$$

Then, we conclude

$$\mu_1\left(\|u_{mx}(t)\|^2\right) \to \mu_1\left(\|u_x(t)\|^2\right) \text{ strongly in } C^0([0,T]). \tag{3.46}$$

Passing to the limit in (3.17) by (3.42), (3.45) and (3.46), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\langle u'(t), v\rangle + \lambda\langle u_x'(t), v_x\rangle + \mu_1\left(\|u_x(t)\|^2\right)\langle u_x(t), v_x\rangle \\ \qquad +\lambda_1\langle u'(t), v\rangle + \sum_{i=1}^N K_i\left\langle |u(t)|^{p_i-2}u(t), v\right\rangle \\ \qquad = \mu_2(t)\int_0^t g(t-s)\langle u_x(s), v_x\rangle\, ds + \langle F(t), v\rangle,\ \forall v\in H_0^1, \text{ a.e., } t\in(0,T), \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{3.47}$$

Theorem 3.4 is proven. □

In the following, to obtain the decay result, besides the assumptions $(\bar{H}_1)$, $(\hat{H}_2)$ for the functions $\tilde{u}_0$, $\tilde{u}_1$, μ_1 as before, the assumptions for the function μ_2, g, F shall be added as follows.

(H_3^∞) : $\mu_2 \in C^1(\mathbb{R}_+)$;
 (i) $\mu_2'(t) \le -\zeta_2\mu_2(t) < 0,\ \ \forall t\ge 0$;
 (ii) $0 < \mu_2(t) \le \mu_2(0),\ \ \forall t \ge 0$;

(H_4^∞) : $g \in L^1(\mathbb{R}_+)$;
 (i) $L = \mu_{1*} - \mu_2(0)\int_0^\infty g(s)ds > 0$;
 (ii) $g'(t) \le -\zeta_1 g(t) < 0,\ \ \forall t \ge 0$;

(H_5^∞) $F \in L^2(\mathbb{R}_+; L^2)$ and there exist two positive constants C_*, η_* such that $\|F(t)\| \le C_* e^{-\eta_* t}$, $\forall t \ge 0$.

Let $(\bar{H}_1)$, $(\hat{H}_2)$, (H_3^∞), (H_4^∞), (H_5^∞) hold and $p_i > 2$, $K_i > 0$ $(i = 1,\cdots,N)$, $\lambda > 0$, $\lambda_1 > 0$. For $T > 0$, based on Theorem 4.2, there exist a unique weak solution u of Prob. (3.11) such that

$$u \in \tilde{W}_1(T) = \{u \in C^0\left([0,T]; H_0^1\right) \cap C^1\left([0,T]; L^2\right) : u' \in L^2(0,T; H_0^1)\}. \tag{3.48}$$

We now construct the Lyapunov functional in the form

$$\mathcal{L}(t) = E(t) + \delta\Psi(t), \tag{3.49}$$

where $\delta > 0$ is chosen suitably and

$$\begin{cases} E(t) = \frac{1}{2}\|u'(t)\|^2 + \sum_{i=1}^N \frac{K_i}{p_i}\|u(t)\|_{L^{p_i}}^{p_i} + \frac{1}{2}\mu_2(t)(g*u)(t) \\ \qquad + \frac{1}{2}\left(\int_0^{\|u_x(t)\|^2}\mu_1(z)\,dz - \mu_2(t)\tilde{g}(t)\|u_x(t)\|^2\right), \\ \Psi(t) = \langle u(t), u'(t)\rangle + \frac{\lambda}{2}\|u_x(t)\|^2 + \frac{\lambda_1}{2}\|u(t)\|^2, \\ (g*u)(t) = \int_0^t g(t-s)\|u_x(t)-u_x(s)\|^2\, ds, \\ \tilde{g}(t) = \int_0^t g(s)ds. \end{cases} \tag{3.50}$$

We have the following estimate for $E'(t)$

Lemma 3.5. *For all $\varepsilon_1 > 0$, we have*

$$E'(t) \leq -\lambda \|u'_x(t)\|^2 - (\lambda_1 - \frac{\varepsilon_1}{2}) \|u'(t)\|^2 - \frac{1}{2}(\zeta_1 + \zeta_2)\mu_2(t)(g * u)(t) \tag{3.51}$$
$$- \frac{1}{2}\mu_2(t) g(t) \|u_x(t)\|^2 - \frac{1}{2}\mu'_2(t) \tilde{g}(t) \|u_x(t)\|^2 + \frac{1}{2\varepsilon_1}\|F(t)\|^2.$$

Proof. . Multiplying $(3.11)_1$ by $u'(x,t)$, and integrating over $[0,1]$, we get

$$E'(t) = -\lambda \|u'_x(t)\|^2 - \lambda_1 \|u'(t)\|^2 + \frac{1}{2}[\mu'_2(t)(g * u)(t) + \mu_2(t)(g' * u)(t)] \tag{3.52}$$
$$- \frac{1}{2}\mu_2(t) g(t) \|u_x(t)\|^2 - \frac{1}{2}\mu'_2(t) \tilde{g}(t) \|u_x(t)\|^2 + \langle F(t), u'(t)\rangle.$$

Using the following inequality

$$\langle F(t), u'(t)\rangle \leq \frac{\varepsilon_1}{2}\|u'(t)\|^2 + \frac{1}{2\varepsilon_1}\|F(t)\|^2, \text{ for all } \varepsilon_1 > 0, \tag{3.53}$$
$$\mu'_2(t)(g * u)(t) \leq -\zeta_2\mu_2(t)(g * u)(t),$$
$$\mu_2(t)(g' * u)(t) \leq -\zeta_1\mu_2(t)(g * u)(t),$$

by (3.52), we obtain (3.51). The lemma is proven. □

We choose $\tilde{\varepsilon}_1 > 0$ such that $0 < \frac{\tilde{\varepsilon}_1}{2} < \lambda_1$. From Lemma 3.5, we deduce

$$E'(t) \leq \frac{1}{2}|\mu'_2(t)| \tilde{g}(t) \|u_x(t)\|^2 + \frac{1}{2\tilde{\varepsilon}_1}\|F(t)\|^2. \tag{3.54}$$

Note that

$$E(0) = \frac{1}{2}\|\tilde{u}_1\|^2 + \sum_{i=1}^{N}\frac{K_i}{p_i}\|\tilde{u}_0\|_{L^{p_i}}^{p_i} + \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|^2}\mu_1(z)\,dz > 0,$$

and

$$E(t) \geq \frac{1}{2}\left(\int_0^{\|u_x(t)\|^2}\mu_1(z)\,dz - \mu_2(t)\tilde{g}(t)\|u_x(t)\|^2\right) \tag{3.55}$$
$$\geq \frac{1}{2}(\mu_{1*} - \mu_2(0)\tilde{g}(\infty))\|u_x(t)\|^2 = \frac{L}{2}\|u_x(t)\|^2,$$

with $\tilde{g}(\infty) = \int_0^\infty g(s)ds$, we have

$$E(t) \leq E(0) + \frac{1}{2\tilde{\varepsilon}_1}\int_0^t \|F(s)\|^2 ds + \frac{1}{2}\int_0^t |\mu'_2(s)| \tilde{g}(s)\|u_x(s)\|^2 ds \tag{3.56}$$
$$\leq E(0) + \frac{1}{2\tilde{\varepsilon}_1}\|F\|^2_{L^2(\mathbb{R}_+;L^2)} + \frac{1}{L}\tilde{g}(\infty)\int_0^t |\mu'_2(s)| E(s)\,ds.$$

From Gronwall's lemma, we get

$$E(t) \leq \left(E(0) + \frac{1}{2\tilde{\varepsilon}_1}\|F\|^2_{L^2(\mathbb{R}_+;L^2)}\right)\exp\left(\frac{1}{L}\tilde{g}(\infty)\int_0^t |\mu'_2(s)|\,ds\right) \tag{3.57}$$
$$\leq \left(E(0) + \frac{1}{2\tilde{\varepsilon}_1}\|F\|^2_{L^2(\mathbb{R}_+;L^2)}\right)\exp\left(\frac{1}{L}\tilde{g}(\infty)\|\mu'_2\|_{L^1(\mathbb{R}_+)}\right) = R_*^2.$$

Consequently, the solution $u(t)$ is expanded and defined over $t \geq 0$. The following lemma states the relation between $\mathcal{L}(t)$ and $E_1(t)$, it is not difficult to prove this lemma.

Lemma 3.6. *We put*

$$E_1(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + \sum_{i=1}^{N} \|u(t)\|_{L^{p_i}}^{p_i} + \mu_2(t)(g * u)(t). \tag{3.58}$$

Then, there exist positive constants $\bar{\beta}_1$, $\bar{\beta}_2$, *such that*

$$\bar{\beta}_1 E_1(t) \le \mathcal{L}(t) \le \bar{\beta}_2 E_1(t), \tag{3.59}$$

with $\delta > 0$ *small enough.*

We have the following estimate for $\Psi'(t)$.

Lemma 3.7. *For all* $\varepsilon_2 > 0$, *we have*

$$\begin{aligned} \Psi'(t) \le \|u'(t)\|^2 &- \left(\mu_{1*} - \frac{\varepsilon_2}{2} - \frac{3}{2}\tilde{g}(\infty)\mu_2(0)\right)\|u_x(t)\|^2 \\ &- \sum_{i=1}^{N} K_i \|u(t)\|_{L^{p_i}}^{p_i} + \frac{1}{2}\mu_2(t)(g * u)(t) + \frac{1}{2\varepsilon_2}\|F(t)\|^2. \end{aligned} \tag{3.60}$$

Proof. Multiplying $(3.11)_1$ by $u(x,t)$, and integrating over $[0,1]$, we get

$$\begin{aligned} \Psi'(t) = \|u'(t)\|^2 &- \mu_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 - \sum_{i=1}^{N} K_i \|u(t)\|_{L^{p_i}}^{p_i} \\ &+ \mu_2(t)\int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle ds + \langle F(t), u(t)\rangle. \end{aligned} \tag{3.61}$$

We note that

$$\begin{aligned} \langle F(t), u(t)\rangle &\le \frac{\varepsilon_2}{2}\|u_x(t)\|^2 + \frac{1}{2\varepsilon_2}\|F(t)\|^2, \text{ for all } \varepsilon_2 > 0, \\ -\mu_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 &\le -\mu_{1*}\|u_x(t)\|^2, \\ \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle ds &\le \frac{1}{2}(g * u)(t) + \frac{3}{2}\tilde{g}(t)\|u_x(t)\|^2. \end{aligned} \tag{3.62}$$

Hence, we obtain (3.60) from (3.61) and (3.62). Lemma 3.7 is proven. □

Finally, we shall have the estimate for the decay of the solution as follows.

Putting $\rho(t) = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2$, and from Lemma 3.5 to Lemma 3.7, we deduce that

$$\begin{aligned} \mathcal{L}'(t) \le &-(\lambda_1 - \frac{\varepsilon_1}{2} - \delta)\|u'(t)\|^2 - \frac{1}{2}(\zeta_1 + \zeta_2 - \delta)\mu_2(t)(g * u)(t) - \delta\sum_{i=1}^{N} K_i \|u(t)\|_{L^{p_i}}^{p_i} \\ &- \left[\delta\left(\mu_{1*} - \frac{\varepsilon_2}{2} - \frac{3}{2}\tilde{g}(\infty)\mu_2(0)\right) - \frac{1}{2}\|\mu_2'\|_{L^\infty}\tilde{g}(\infty)\right]\|u_x(t)\|^2 + \rho(t), \end{aligned} \tag{3.63}$$

for all δ, $0 < \delta < \min\{1, \frac{L}{2}\}$, $\forall \varepsilon_1 > 0$ and $\forall \varepsilon_2 > 0$.

We choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\lambda_1 - \frac{\varepsilon_1}{2} > 0,\ \mu_{1*} - \frac{\varepsilon_2}{2} > 0. \tag{3.64}$$

We continue by choosing $\delta > 0$ such that

$$\begin{aligned} 0 < \delta < \min\{1, \frac{L}{2}\}, \quad \theta_1 = \lambda_1 - \frac{\varepsilon_1}{2} - \delta > 0, \\ 2\theta_2 = \zeta_1 + \zeta_2 - \delta > 0. \end{aligned} \tag{3.65}$$

From the assumption of F, we have

$$\rho(t) = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2 \le \tilde{C}_1 e^{-2\eta_* t},\ \forall t \ge 0, \tag{3.66}$$

where $\tilde{C}_1 = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right) C_*^2$.

We can choose $\tilde{g}(\infty) = \int_0^\infty g(s)ds > 0$ is small enough such that

$$\theta_3 \equiv \delta\left(\mu_{1*} - \frac{\varepsilon_2}{2} - \frac{3}{2}\tilde{g}(\infty)\mu_2(0)\right) - \frac{1}{2}\|\mu_2'\|_{L^\infty}\tilde{g}(\infty) > 0, \tag{3.67}$$

it leads to

$$0 < \tilde{g}(\infty) = \int_0^\infty g(s)ds < \frac{2\delta\left(\mu_{1*} - \frac{\varepsilon_2}{2}\right)}{3\delta\mu_2(0) + \|\mu_2'\|_{L^\infty}}. \tag{3.68}$$

Then

$$\begin{aligned} \mathcal{L}'(t) &\le -\theta_1\|u'(t)\|^2 - \theta_2\mu_2(t)(g*u)(t) - \delta\sum_{i=1}^N K_i\|u(t)\|_{L^{p_i}}^{p_i} - \theta_3\|u_x(t)\|^2 + \tilde{C}_1 e^{-2\eta_* t} \\ &\le -\beta_3 E_1(t) + \tilde{C}_1 e^{-2\eta_* t} \le -\frac{\beta_3}{\bar{\beta}_2}\mathcal{L}(t) + \tilde{C}_1 e^{-2\eta_* t} \le -\bar{\gamma}\mathcal{L}(t) + \tilde{C}_1 e^{-2\eta_* t}, \end{aligned}$$

where $\beta_3 = \min\{\theta_1, \theta_2, \theta_3, \delta K, \delta K_1\}$ and $0 < \bar{\gamma} < \min\left\{2\eta_*, \frac{\beta_3}{\bar{\beta}_2}\right\}$.

It gives

$$\mathcal{L}(t) \le \left(\mathcal{L}(0) + \frac{\tilde{C}_1}{2\eta_* - \bar{\gamma}}\right)e^{-\bar{\gamma}t}. \tag{3.69}$$

Using Lemma 3.6, and from (3.69), we obtain

$$E_1(t) \le \frac{1}{\bar{\beta}_1}\mathcal{L}(t) \le \frac{1}{\bar{\beta}_1}\left(\mathcal{L}(0) + \frac{\tilde{C}_1}{2\eta_* - \bar{\gamma}}\right)e^{-\bar{\gamma}t} \equiv \bar{C}e^{-\bar{\gamma}t}, \quad \forall t \ge 0. \tag{3.70}$$

Therefore, we obtain the main result in this section as follows.

Theorem 3.8. *Assume that* $\left(\bar{H}_1\right)$, $(\hat{H}_2)$, (H_3^∞), (H_4^∞), *and* (H_5^∞) *hold and* $\lambda_1 > 0$, $\lambda > 0$, $p_i > 2$, $K_i > 0$ $(i = 1, \cdots, N)$. *Let* $\int_0^\infty g(s)ds > 0$ *be small enough, then there exists a unique weak solution* u *of Prob.* (3.11) *such that* u *belongs to* $C^0\left(\mathbb{R}_+; H_0^1\right) \cap C^1\left(\mathbb{R}_+; L^2\right)$. *Furthermore, there exist positive constants* $\bar{C}$, $\bar{\gamma}$, *such that*

$$\|u'(t)\|^2 + \|u_x(t)\|^2 + \sum_{i=1}^{N} \|u(t)\|_{L^{p_i}}^{p_i} \le \bar{C} e^{-\bar{\gamma} t}, \quad \forall t \ge 0. \tag{3.71}$$

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