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High-order iterative scheme to the Robin problem for a nonlinear wave equation with viscoelastic term

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ABSTRACT
The report deals with the Robin problem for a nonlinear wave equation with viscoelastic term. Under some suitable conditions, we establish a high-order iterative scheme and then prove that the scheme converges to the weak solution of the original problem along with the error estimate. This result extends the result in [9].
Keywords: *Faedo-Galerkin method, High-order iterative scheme, Nonlinear wave equation, Local existence.*

1 Introduction

This report is devoved to study the Robin problem for a nonlinear wave equation with viscoelastic term as follows

$$\begin{cases} u_{tt} - u_{xx} + \lambda(x,t,u)\left|u_t\right|^{q-2} u_t + \int_0^t g(t-s) u_{xx}(x,s)ds = f(x,t,u), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - u(0,t) = u_x(1,t) + u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $q \geq 2$ is a given constant and λ, f, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions with $\lambda(x,t,u) \geq \lambda_* > 0$.

Equation $(1.1)_1$ usually arises within frameworks of mathematical models in engineering and physical sciences. The left-hand integral of equation $(1.1)_1$ is called viscoelastic term.

When $\lambda(x,t,u) \equiv a$, $g = 0$ and $f \equiv b\left|u\right|^{p-2} u$, equation $(1.1)_1$ becomes the following nonlinear wave equation

$$u_{tt} - \Delta u + a\left|u_t\right|^{q-2} u_t = b\left|u\right|^{p-2} u, \tag{1.2}$$

where a, $b > 0$ and p, $q \geq 2$. This equation has been widely studied and obtained many interesting results such as the global existence, exponential decay and finite-time blow-up of solutions (see [1], [2], [4], [10], [12]).

When $\lambda(x,t,u) \equiv 1$ and $f \equiv b\,|u|^{p-2}\,u$, equation $(1.1)_1$ is reduced to the viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{q-2}\,u_t = |u|^{p-2}\,u, \tag{1.3}$$

this form was considered by Messaoudi in [6], where the author proved a finite-time blow-up result for solutions with negative initial energy if $p > q$ and a global existence result for $q \geq p$. Laterly, Kafini and Messaoudi [3] also obtained a blow-up result of a Cauchy problem for a nonlinear viscoelastic equation in the form (1.3) with $q = 2$.

In this paper, we associate with equation $(1.1)_1$ a recurrent sequence $\{u_m\}$ defined by

$$\begin{cases} u_0 \equiv 0, \\ u_m'' - \Delta u_m + \lambda(x,t,u_m)\,|u_m'|^{q-2}\,u_m' + \displaystyle\int_0^t g(t-s)\Delta u_m(s)ds \\ \qquad = \displaystyle\sum_{i=0}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})\,(u_m - u_{m-1})^i\,,\ 0<x<1,\ 0<t<T, \\ u_{mx}(0,t) - u_m(0,t) = u_{mx}(1,t) + u_m(1,t) = 0, \\ u_m(x,0) = \tilde{u}_0(x),\ u_{mt}(x,0) = \tilde{u}_1(x),\ m = 1,2,\cdots. \end{cases} \tag{1.4}$$

If $\lambda \in C^1\left([0,1]\times[0,T^*]\times\mathbb{R}\right),\ \lambda(x,t,u) \geq \lambda_* > 0,\ g \in H^1\left(0,T^*\right),\ f \in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R})$ and some other conditions, we prove that the sequence $\{u_m\}$ converges at the N-order rate to the unique weak solution of Prob. (1.1), it means that

$$\|u_m - u\|_X \leq C\,\|u_{m-1} - u\|_X^N\,, \tag{1.5}$$

for some $C > 0$, where X is a suitable space. The scheme (1.4) is called the high-order iterative scheme or the N-order iterative scheme. We note more that the high-order iterative schemes as above were also used to obtain the existence of solutions in the previous papers, for example, see [7], [8], [9], [11].

This paper consists of four sections. Section 2 is devoted to the presentation of preliminaries. In Section 3, by using the Faedo-Galerkin approximation method and the arguments of compactness, we prove Theorem 3.1 to get the high-order iterative scheme (1.4). Finally, in Section 4, we prove Theorem 4.1 to obtain the convergence of the high-order iterative scheme (1.4) and then, the unique existence of a weak solution of Prob. (1.1) follows. The result obtained here is a generalization of the results of [9] and based on the ideas about recurrence relations as in [7], [8], [9], [11].

2 Preliminaries

Put $\Omega = (0,1)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and $\|\cdot\|_X$ is the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup}\|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

We write $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, to denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$; $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le k$, $D^{(0,0,0)} f = D^{(0)} f = f$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

We also define the following bilinear form and the other norms on H^1

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + u(0)v(0) + u(1)v(1), \ \forall u, v \in H^1, \tag{2.1}$$

$$\|v\|_a = \sqrt{a(v,v)}, \ \forall v \in H^1, \tag{2.2}$$

and

$$\|v\|_i = \left(v^2(i) + \int_0^1 v_x^2(x) dx\right)^{1/2}, \ i = 0, 1. \tag{2.3}$$

On H^1, three norms $\|v\|_{H^1}$, $\|v\|_a$ and $\|v\|_i$ are equivalent norms.

We now have the following lemmas, the proofs of which are straighforward so we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\bar{\Omega})$ is compact and*

$$\begin{array}{ll} \text{(i)} & \|v\|_{C^0(\bar{\Omega})} \le \sqrt{2} \|v\|_{H^1}, \\ \text{(ii)} & \|v\|_{C^0(\bar{\Omega})} \le \sqrt{2} \|v\|_i, \\ \text{(iii)} & \frac{1}{\sqrt{3}} \|v\|_{H^1} \le \|v\|_i \le \sqrt{3} \|v\|_{H^1}, \end{array} \tag{2.4}$$

for all $v \in H^1$, $i = 0, 1$.

Lemma 2.2. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.1) is continuous on $H^1 \times H^1$ and coercive on H^1*, i.e.,

$$\begin{array}{ll} \text{(i)} & |a(u,v)| \le 5 \|u\|_{H^1} \|v\|_{H^1}, \text{ for all } u, v \in H^1, \\ \text{(ii)} & a(u,u) \ge \dfrac{1}{3} \|u\|_{H^1}^2, \text{ for all } u \in H^1. \end{array} \tag{2.5}$$

3 Main results

3.1 A high-order iterative scheme

In this section, we shall establish a high-order iterative scheme in order to obtain the existence of a weak solution for Prob. (1.1). Let us note here that the weak solution u of Prob. (1.1) will be obtained in Section 4 (Theorem 4.1) in the following manner:

Find $u \in L^\infty(0,T;H^2)$ such that $u' \in L^\infty(0,T;H^1)$, $u'' \in L^\infty(0,T;L^2)$ and u satisfies the following variational problem and the initial conditions

$$\begin{cases} \langle u''(t), w \rangle + a(u(t), w) + \left\langle \lambda(t, u(t)) \left|u'(t)\right|^{q-2} u'(t), w \right\rangle \\ \qquad\qquad = \displaystyle\int_0^t g(t-s) a(u(s), w) ds + \langle f(x,t,u), w \rangle, \ \forall w \in H^1, \\ u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \end{cases} \tag{3.1}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on H^1 defined by (2.1).

Let $T^* > 0$, we make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$;
(H_2) $g \in H^1(0, T^*)$;
(H_3) $\lambda \in C^1([0,1] \times [0,T^*] \times \mathbb{R})$, and there exists a positive constant λ_* such that $\lambda(x,t,u) \geq \lambda_* > 0,\ \forall (x,t,u) \in [0,1] \times [0,T^*] \times \mathbb{R}$;
(H_4) $f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$ such that
(i) $D_3^i f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $1 \leq i \leq N$,
(ii) $D_1 D_3^i f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $0 \leq i \leq N-1$.

Fix $T^* > 0$. For each $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;H^2) : v' \in L^\infty(0,T;H^1),\ v'' \in L^2(Q_T), \\ \qquad\qquad \text{with } \|v\|_{L^\infty(0,T;H^2)},\ \|v'\|_{L^\infty(0,T;H^1)},\ \|v''\|_{L^2(Q_T)} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) :\ v'' \in L^\infty(0,T;L^2)\}. \end{cases} \tag{3.2}$$

Now, we construct the following recurrent sequence $\{u_m\}$:
The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.3}$$

we find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the nonlinear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a(u_m(t), w) + \langle \lambda(t, u_m(t)) \left|u_m'(t)\right|^{q-2} u_m'(t), w\rangle \\ \qquad = \displaystyle\int_0^t g(t-s) a(u_m(s), w) ds + \langle F_m(t), w\rangle,\ \forall w \in H^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.4}$$

in which

$$F_m(x,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}) \left(u_m - u_{m-1}\right)^i. \tag{3.5}$$

Then we have the following theorem.

Theorem 3.1. *Let $(H_1)-(H_4)$ hold. Then there exist a constant $M > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$ and a constant $T > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$, g, f, q and λ such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by (3.4)-(3.5).*

Proof. The proof is based on the Faedo - Galerkin approximation method introduced by Lions [5], the arguments of compactness, together with the same evaluation techniques as in [9]. □

3.2 Convergence and error estimate of the scheme

This section is devoted to prove the N-order convergence of the sequence $\{u_m\}$ established in Theorem 3.1 to the weak solution of Prob. (1.1). First, we denote

$$W_1(T) = C([0,T]; H^1) \cap C^1([0,T]; L^2), \tag{3.6}$$

it is clear to see that $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{C([0,T];H^1)} + \|v'\|_{C^0([0,T];L^2)}. \tag{3.7}$$

Then we have the following theorem.

Theorem 3.2. *Let $(H_1)-(H_4)$ hold. Then, there exist constants $M>0$ and $T>0$ defined as in Theorem 3.1 such that*

(i) Prob. (1.1) has a unique weak solution $u \in W_1(M,T)$ and the sequence $\{u_m\}$ defined by (3.4)-(3.5) converges at a rate of order N to the solution u strongly in the space $W_1(T)$, in the sense

$$\|u_m - u\|_{W_1(T)} \leq C\,\|u_{m-1} - u\|_{W_1(T)}^N, \tag{3.8}$$

for all $m \geq 1$, where C is a suitable constant.

(ii) Furthermore, the following estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \leq C_T\,(\gamma_T)^{N^m}, \text{ for all } m \in \mathbb{N}, \tag{3.9}$$

where C_T and $0<\gamma_T<1$ are the constants depending only on T.

Proof. (i) *Existence of a solution.* We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Indeed, we put $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + a\left(v_m(t), w\right) \\ \qquad +\langle \lambda(t, u_{m+1}(t)) \left[\left|u_{m+1}'(t)\right|^{q-2} u_{m+1}'(t), w\rangle - \left|u_m'(t)\right|^{q-2} u_m'(t)\right], w\rangle \\ \qquad = -\langle [\lambda(t,u_{m+1}(t)) - \lambda(t,u_m(t))]\,|u_m'(t)|^{q-2}\,u_m'(t), w\rangle \\ \qquad\quad + \displaystyle\int_0^t g(t-s) a\left(v_m(s), w\right) ds + \langle F_{m+1}(t) - F_m(t), w\rangle,\ \forall w \in H^1, \\ v_m(0) = v_m'(0) = 0. \end{cases} \tag{3.10}$$

Taking $w = v_m'$ in (3.10), after integrating in t, and noting that

$$-2\int_0^t \left\langle \lambda(s,u_{m+1}(s)) \left(\left|u_{m+1}'(s)\right|^{q-2} u_{m+1}'(s) - \left|u_m'(s)\right|^{q-2} u_m'(s)\right), v_m'(s)\right\rangle ds \leq 0,$$

we get

$$\begin{aligned} X_m(t) \leq & -2\int_0^t \langle [\lambda(s,u_{m+1}(s)) - \lambda(s,u_m(s))]\,|u_m'(s)|^{q-2}\,u_m'(s), v_m'(s)\rangle ds \\ & + 2\int_0^t g(t-\tau) a\left(v_m(\tau), v_m(t)\right) d\tau - 2\int_0^t g(0)\, a\left(v_m(s), v_m(s)\right) ds \\ & - 2\int_0^t ds \int_0^s g'(s-\tau) a\left(v_m(\tau), v_m(s)\right) d\tau \\ & + 2\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle\, ds \\ \equiv & \sum_{k=1}^5 J_k, \end{aligned} \tag{3.11}$$

with

$$X_m(t) = \|v_m'(t)\|^2 + \|v_m(t)\|_a^2. \tag{3.12}$$

We denote the constants $K_M(f)$, $\bar{K}_M(\lambda)$, as follows

$$\begin{cases} K_M(f) = \|f\|_{C^0(\Omega_M)} + \sum_{i=1}^N \|D_3^i f\|_{C^0(\Omega_M)} + \sum_{i=1}^{N-1} \|D_1 D_3^i f\|_{C^0(\Omega_M)}, \\ \|f\|_{C^0(\Omega_M)} = \sup_{(x,t,u)\in\Omega_M} |f(x,t,u)|, \\ \bar{K}_M(\lambda) = \|D_3\lambda\|_{C^0(\Omega_M)}, \\ \Omega_M = [0,1]\times[0,T^*]\times[-\sqrt{2}M, \sqrt{2}M]. \end{cases} \tag{3.13}$$

Next, we need to estimate the integrals on the right side of (3.11) as follows.

First, it is not difficult to estimate terms J_1, J_2, J_3 and J_4 as follows:

$$\begin{aligned} J_1 &= -2\int_0^t \langle [\lambda(s,u_{m+1}(s)) - \lambda(s,u_m(s))] \left|u_m'(s)\right|^{q-2} u_m'(s), v_m'(s)\rangle ds \\ &\le 2\bar{K}_M(\lambda)M^{q-1}\int_0^t \|v_m(s)\| \left\|v_m'(s)\right\| ds \le \bar{K}_M(\lambda)M^{q-1}\int_0^t X_m(s)ds; \\ J_2 &= 2\int_0^t g(t-\tau)a\left(v_m(\tau), v_m(t)\right)d\tau \le \frac{1}{2}X_m(t) + 2\left\|g\right\|^2_{L^2(0,T^*)}\int_0^t X_m(s)ds; \\ J_3 &= -2\int_0^t g\left(0\right)a\left(v_m(s), v_m(s)\right)ds \le 2\left|g\left(0\right)\right|\int_0^t X_m(s)ds; \\ J_4 &= -2\int_0^t ds\int_0^s g'(s-\tau)a\left(v_m(\tau), v_m(s)\right)d\tau \le 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t X_m(s)ds. \end{aligned} \tag{3.14}$$

Next, using Taylor's expansion of the function $f(x,t,u_m) = f(x,t,u_{m-1}+v_{m-1})$ around the point u_{m-1} up to order N, we obtain

$$f(x,t,u_m) - f(x,t,u_{m-1}) = \sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_{m-1})v_{m-1}^i + \frac{1}{N!}D_3^N f(x,t,\tilde{\theta}_m)v_{m-1}^N, \tag{3.15}$$

where $\tilde{\theta}_m = \tilde{\theta}_m(x,t) = u_m + \theta_1 v_{m-1}$, $0 < \theta_1 < 1$.

Hence, it follows from (3.5) and (3.15) that

$$F_{m+1}(x,t) - F_m(x,t) = \sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_m)v_m^i + \frac{1}{N!}D_3^N f(x,t,\tilde{\theta}_m)v_{m-1}^N. \tag{3.16}$$

Therefore, we have

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\| &\le K_M(f)\sum_{i=1}^{N-1}\frac{1}{i!}(\sqrt{2}\left\|v_m(t)\right\|_{H^1})^i + \frac{1}{N!}K_M(f)(\sqrt{2}\left\|v_{m-1}(t)\right\|_{H^1})^N \\ &\le \beta_T^{(1)}\sqrt{X_m(t)} + \beta_T^{(2)}\left\|v_{m-1}\right\|^N_{W_1(T)}, \end{aligned} \tag{3.17}$$

where $\beta_T^{(1)} = \sqrt{6}K_M(f)\sum_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{2}M\right)^{i-1}$, $\beta_T^{(2)} = \frac{\sqrt{2}^N}{N!}K_M(f)$.

It implies that

$$\begin{aligned} J_5 &= 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), v_m'(s)\right\rangle ds \\ &\le 2\int_0^t \|F_{m+1}(s) - F_m(s)\| \left\|v_m'(s)\right\| ds \\ &\le 2\int_0^t \left(\beta_T^{(1)}\sqrt{X_m(s)} + \beta_T^{(2)}\left\|v_{m-1}\right\|^N_{W_1(T)}\right)\sqrt{X_m(s)}ds \\ &\le 2\beta_T^{(1)}\int_0^t X_m(s)ds + 2\beta_T^{(2)}\left\|v_{m-1}\right\|^N_{W_1(T)}\int_0^t \sqrt{X_m(s)}ds \\ &\le 2\beta_T^{(1)}\int_0^t X_m(s)ds + T\beta_T^{(2)}\left\|v_{m-1}\right\|^{2N}_{W_1(T)} + \beta_T^{(2)}\int_0^t X_m(s)ds. \end{aligned} \tag{3.18}$$

Combining (3.11), (3.14) and (3.18), we obtain

$$X_m(t) \leq 2T\beta_T^{(2)} \|v_{m-1}\|_{W_1(T)}^{2N} + \beta_T^{(3)} \int_0^t X_m(s)ds, \tag{3.19}$$

where
$\beta_T^{(3)} = 2\left[\bar{K}_M(\lambda)M^{q-1} + 2\left(|g(0)| + \|g\|^2_{L^2(0,T^*)} + \sqrt{T^*}\|g'\|_{L^2(0,T^*)} + \beta_T^{(1)}\right) + \beta_T^{(2)}\right]$.

By using Gronwall's lemma, (3.19) gives

$$\|v_m\|_{W_1(T)} \leq \mu_T \|v_{m-1}\|^N_{W_1(T)}, \tag{3.20}$$

with $\mu_T = \left(1+\sqrt{3}\right)\sqrt{2T\beta_T^{(2)}\exp(T\beta_T^{(3)})}$.

Choosing $T>0$ small enough such that $\gamma_T = M\mu_T^{\frac{1}{N-1}} < 1$, it follows from (3.20) that

$$\|u_m - u_{m+p}\|_{W_1(T)} \leq (1-\gamma_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}(\gamma_T)^{N^m}, \text{ for all } m \text{ and } p \in \mathbb{N}. \tag{3.21}$$

Hence, $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Thus, there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.22}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^1) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u \in W(M,T). \end{cases} \tag{3.23}$$

Moreover, by (3.22) and the inequalities

$$\sup_{0\leq t\leq T}\|\lambda(t,u_m(t)) - \lambda(t,u(t))\| \leq \bar{K}_M(\lambda)\|u_m - u\|_{W_1(T)}, \tag{3.24}$$

$$\left\| |u'_m|^{q-2}u'_m - |u'|^{q-2}u'\right\|_{C^0([0,T];L^2)} \leq (q-1)\left(\sqrt{2}M\right)^{q-2}\|u_m - u\|_{W_1(T)},$$

we have

$$\lambda(\cdot,t,u_m(t)) \to \lambda(\cdot,t,u(t)) \text{ strongly in } C^0([0,T];L^2), \tag{3.25}$$
$$|u'_m|^{q-2}u'_m \to |u'|^{q-2}u' \text{ strongly in } C^0([0,T];L^2).$$

On the other hand

$$\begin{aligned} &\|F_m(\cdot,t) - f(\cdot,t,u(t))\| \\ &\leq \|f(\cdot,t,u_{m-1}(t)) - f(\cdot,t,u(t))\| + \left\|\sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(\cdot,t,u_{m-1})(u_m - u_{m-1})^i\right\| \\ &\leq K_M(f)\left[\|u_{m-1} - u\|_{W_1(T)} + \sum_{i=1}^{N-1}\frac{1}{i!}\|u_m - u_{m-1}\|^i_{W_1(T)}\right]. \end{aligned} \tag{3.26}$$

Therefore, it implies from (3.22) and (3.25) that

$$F_m(t) \to f(\cdot, t, u(t)) \text{ strongly in } C^0([0,T]\,; L^2). \tag{3.27}$$

Finally, passing to limit in (3.4) and (3.5) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\begin{aligned} &\langle u''(t), w\rangle + a(u(t), w) + \left\langle \lambda(t, u(t)) \left|u'(t)\right|^{q-2} u'(t), w \right\rangle \\ &= \int_0^t g(t-s)a(u(s), w)ds + \langle f(\cdot, t, u(t)), w\rangle, \end{aligned} \tag{3.28}$$

for all $w \in H^1$ and the initial condition

$$u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1. \tag{3.29}$$

On the other hand, it follows from $(3.23)_4$ and (3.28) that

$$u'' = \Delta u - \lambda(x,t,u) \left|u'\right|^{q-2} u' + \int_0^t g(t-s)\Delta u(s)ds + f(x,t,u) \in L^\infty(0,T; L^2), \tag{3.30}$$

hence, $u \in W_1(M,T)$.

Uniqueness. Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1). Then $\bar{u} = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle \bar{u}''(t), w\rangle + a(\bar{u}(t), w) \\ \qquad = -\left\langle \lambda(t, u_1(t)) \left(\left|u_1'(t)\right|^{q-2} u_1'(t) - \left|u_2'(t)\right|^{q-2} u_2'(t)\right), w \right\rangle \\ \qquad\; - \left\langle (\lambda(t, u_1(t)) - \lambda(t, u_2(t))) \left|u_2'(t)\right|^{q-2} u_2'(t), w \right\rangle \\ \qquad\;\; + \displaystyle\int_0^t g(t-s)a(\bar{u}(s), w)ds + \langle f(x,t,u_1) - f(x,t,u_2), w\rangle, \;\; \forall w \in H^1, \\ \bar{u}(0) = \bar{u}'(0) = 0. \end{cases} \tag{3.31}$$

We take $w = \bar{u}'(t)$ in $(3.31)_1$ and integrate in t to get

$$\begin{aligned} \rho(t) &= \left\|\bar{u}'(t)\right\|^2 + \left\|\bar{u}(t)\right\|_a^2 \qquad (3.32) \\ &\leq -2\int_0^t \left\langle (\lambda(s, u_1(s)) - \lambda(s, u_2(s))) \left|u_2'(s)\right|^{q-2} u_2'(s), \bar{u}'(s) \right\rangle ds \\ &\quad + 2\int_0^t g(t-\tau)a\left(\bar{u}(\tau), \bar{u}(t)\right) d\tau - 2\int_0^t g(0)\, a\left(\bar{u}(s), \bar{u}(s)\right) ds \\ &\quad - 2\int_0^t ds \int_0^s g'(s-\tau)a\left(\bar{u}(\tau), \bar{u}(s)\right) d\tau \qquad (3.33) \\ &\quad + 2\int_0^t \left\langle f(x,s,u_1(s)) - f(x,s,u_2(s)), \bar{u}'(s)\right\rangle ds \\ &\equiv \sum_{k=1}^4 \bar{J}_k, \end{aligned}$$

We estimate the integrals $\bar{J}_k,\ k=\overline{1,5}$ as follows.

$$\begin{aligned}\bar{J}_1 &= -2\int_0^t \left\langle \left(\lambda(s,u_1(s)) - \lambda(s,u_2(s))\right)\left|u_2'(s)\right|^{q-2} u_2'(s), \bar{u}'(s)\right\rangle ds \\ &\le 2\bar{K}_M(\lambda)M^{q-1}\int_0^t \|\bar{u}(s)\|\,\|\bar{u}'(s)\|\,ds \le \bar{K}_M(\lambda)M^{q-1}\int_0^t \rho(s)ds; \\ \bar{J}_2 &= 2\int_0^t g(t-\tau)a\left(\bar{u}(\tau),\bar{u}(t)\right)d\tau \le \frac{1}{2}\rho(t) + 2\|g\|^2_{L^2(0,T^*)}\int_0^t \rho(s)ds; \\ \bar{J}_3 &= -2\int_0^t g(0)\,a\left(\bar{u}(s),\bar{u}(s)\right)ds \le 2|g(0)|\int_0^t \rho(s)ds; \\ \bar{J}_4 &= -2\int_0^t ds\int_0^s g'(s-\tau)a\left(\bar{u}(\tau),\bar{u}(s)\right)d\tau \le 2\sqrt{T^*}\,\|g'\|_{L^2(0,T^*)}\int_0^t \rho(s)ds; \\ \bar{J}_5 &= 2\int_0^t \left\langle f(x,s,u_1(s)) - f(x,s,u_2(s)), \bar{u}'(s)\right\rangle ds \le 2\sqrt{6}K_M(f)\int_0^t \rho(s)ds.\end{aligned} \tag{3.34}$$

We deduce from (3.32) and (3.34), that

$$\rho(t) = \|\bar{u}'(t)\|^2 + \|\bar{u}(t)\|_a^2 \le k_T\int_0^t \rho(s)ds, \tag{3.35}$$

where

$$k_T = 2\left[\bar{K}_M(\lambda)M^{q-1} + 2\left(\|g\|^2_{L^2(0,T^*)} + |g(0)| + \sqrt{T^*}\,\|g'\|_{L^2(0,T^*)} + \sqrt{6}K_M(f)\right)\right].$$

Using Gronwall's Lemma, it follows that $\rho(t) = \|\bar{u}'(t)\|^2 + \|\bar{u}(t)\|_a^2 \equiv 0$, i.e., $\bar{u} = u_1 - u_2 = 0$. Therefore, $u \in W_1(M,T)$ is an unique local weak solution of Prob. (1.1).

(ii) Passing to the limit in (3.21) as $p \to \infty$ for fixed m, we get (3.9).

By the similar argument, (3.8) follows. Theorem 3.2 is proved completely. □

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