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On the Robin-Dirchlet problem for a nonlinear wave equation with the term $\frac{1}{n}\sum_{i=1}^{n} u^2(\frac{i-1}{n}, t)$

by ***Nguyen Vu Dzung*** *(Faculty of Mathematics and Computer Science, VNUHCM-University of Science, Viet nam; Cao Thang Technical College),* ***Tran Trinh Manh Dung*** *(Faculty of Mathematics and Computer Science, VNUHCM- University of Science, Viet nam; Division of Mathematics, FPT University, Viet nam)*

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Corresponding author:dzungngv@gmail.com

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ABSTRACT

In this paper, we study the Robin-Dirchlet problem (P_n) for a wave equation with the term $\frac{1}{n}\sum_{i=1}^{n} u^2(\frac{i-1}{n}, t),\ n \in \mathbb{N}$. First, for each $n \in \mathbb{N}$, under suitable conditions, we prove the local existence and uniqueness of the weak solution u^n of (P_n). Next, we prove that the sequence of solutions u^n of (P_n) converges strongly in appropriate spaces to the weak solution u of the problem (P), where (P) is defined by (P_n) by replacing $\frac{1}{n}\sum_{i=1}^{n} u^2(\frac{i-1}{n}, t)$ by $\int_0^1 u^2(y,t)dy$. The main tools used here are the linearization method together with the Faedo-Galerkin method and the weak compact method. We end the paper with a remark related to a similar problem.

Keywords: *Robin-Dirichlet problem, Faedo-Galerkin method, linearization method, weak compact method.*

1 Introduction

In this paper, we study the Robin-Dirichlet problem for a wave equation as follows

$$(P_n) \begin{cases} u_{tt} - \left(1 + (\bar{S}_n u)(t)\right) u_{xx} = f\left(x, t, u, u_x, u_t\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\zeta \geq 0$ is a given constant and

$$(\bar{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u^2(\tfrac{i-1}{n}, t),\ n \in \mathbb{N}. \tag{1.2}$$

The idea of considering Prob. (1.1) comes from the study of initial boundary value problems for the wave equation in the form

$$u_{tt} - \left(1 + \int_0^1 u^2(y,t)dy\right) u_{xx} = f\left(x,t,u,u_x,u_t\right), \tag{1.3}$$

associted with the initial boundary conditions $(1.1)_{2-3}$. Note that, Eq. (1.3) with the case $f = 0$ is related to Carrier-type equations. Precisely, Carrier [1] studied the vibration of an elastic string when the changes in tension are not small, of which the model was constructed in the form

$$\rho h u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2 dx\right) u_{xx} = 0, \tag{1.4}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of a string, ρ is the density of a material. When $f = 0$, and replacing $\int_0^1 u^2(y,t)dy$ by $\int_0^1 u_x^2(y,t)dy$, the Eq. (1.3) is known as Kirchhoff-type equations that the original equation was introduced by Kirchhoff [6]

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L u_x^2(y,t)dy\right) u_{xx}, \tag{1.5}$$

decribing free vibrations of elastic strings taking into account the changes in length of the string produced by transverse vibrations, where u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. For many decades, numerous of studies about Kirchhoff-Carrier type equation have been published, for example, we refer to the works of Cavalcanti et. al. [2]- [4], in which the results of global existence, exponential, uniform decay rates, and asymptotic behavior for the different models of Kirchhoff-type equations have been obtained. Indeed, Cavalcanti et. al. [2] proved the existence of global solutions and exponential decay for the following nonlinear problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla y|^2 dx\right)\Delta y - \dfrac{\partial}{\partial t}\Delta y = f, \text{ in } Q = \Omega \times (0,\infty), \\ y = 0, \text{ on } \Sigma_1 = \Gamma_1 \times (0,\infty), \\ M\left(\displaystyle\int_\Omega |\nabla y|^2 dx\right)\dfrac{\partial y}{\partial \nu} + \dfrac{\partial}{\partial t}\left(\dfrac{\partial y}{\partial \nu}\right) = g, \text{ on } \Sigma_0 = \Gamma_0 \times (0,\infty), \\ y(0) = y^0,\ \dfrac{\partial y}{\partial t}(0) = y^1, \text{ in } \Omega, \end{cases} \tag{1.6}$$

where M is a C^1-function and $M(\lambda) \geq \lambda_0 > 0$, $\forall \lambda \geq 0$. Naturally, some studies paid attention to extending in mathematical context of Kirchhoff-Carrier equations, see in [7], [9], [12], [15], [16], [18], [19]. In [7], Long proved the solvability and established an asymptotic expansion in a small parameter of solutions for a nonlinear wave equation of Kirchhoff-Carrier type as follows

$$\begin{cases} u_{tt} - B\left(t, \|u\|^2, \|u_x\|^2\right) u_{xx} = f\left(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2\right), \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.7}$$

where $h_0 > 0$, $h_1 \geq 0$ are constants and B, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, afterward, these results were extended in [16]. In [19], Yang and Gong established the blow-up of solutions for the following viscoelastic equation of Kirchhoff type

$$u_{tt}(x,t) - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1}u, \tag{1.8}$$

associated with Dirichlet boundary conditions. We also refer to [13] and [14] for some various results of blow-up phenomenon of solutions for the Kirchhoff-type equations.

On the other hand, the idea for considering Prob. (1.1) also comes from the study of problems containing the partitions of domain, for example, see [5], [11], [17] and the references therein. In [5], Il'in established the uniqueness theorem for generalized solution to the mixed problem for a wave equation with nonlocal boundary conditions (also called multi-point type conditions)

$$\begin{cases} u_{tt} - u_{xx} = 0,\ 0 < x < L,\ 0 < t < T, \\ u(0,t) = \mu(t),\ u(L,t) = \sum_{i=1}^{k} \alpha_i(t)u(\xi_i,t),\ 0 < t < T, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x),\ 0 < x < L, \end{cases} \tag{1.9}$$

where $\xi_1, \cdots, \xi_k$ are positive constants, $0 \leq \xi_1 < \cdots < \xi_k < L$, and $\alpha_i(t)$, $i = 1, \cdots, k$ are given functions. In [11], Nhan et. al. considered the Robin problem for a nonlinear wave equation with source containing multi-point nonlocal terms as follows

$$\begin{cases} u_{tt} - u_{xx} = f\left(x, t, u(x,t), u(\eta_1,t), \cdots, u(\eta_q,t), u_t(x,t)\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.10}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and h_0, $h_1 \geq 0$, $\eta_1, \cdots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leq \eta_1 < \cdots < \eta_q \leq 1$. Here, we note more that, if we put $(\bar{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u^2(\frac{i-1}{n}, t)$, then $(\bar{S}_n u)(t)$ can be considered as a combination of the unknown values $u(\eta_1,t), \cdots, u(\eta_q,t)$ of Prob. (1.10) in the case of $q = n$ and $\eta_i = \frac{i-1}{n}$.

Motivated by the above mentioned works and based on the recent result of Ngoc et. al. [10] for a strongly damped wave equation with arithmetic-mean terms, because of mathematical context, we study the existence and some properties of the solution for Prob. (1.1).

Let us explain in some detail which are our main results. First, for each $n \in \mathbb{N}$ fixed, we prove the existence and uniqueness of a local weak solution u^n of Prob. (P_n). Then, we can consider the behavior of solutions u^n, $\forall n \in \mathbb{N}$. It is clear to see that, if $u \in L^\infty(0,T;H^1)$ then the function $y \longmapsto u^2(y,t)$ is continuous on $[0,1]$, a.e. $t \in [0,T]$, it leads to

$$(\bar{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u^2(\tfrac{i-1}{n}, t) \to \int_0^1 u^2(x,t)dx = \|u(t)\|^2,\ \text{as } n \to \infty.$$

Therefore, it is possible that Prob. (P_n) have a close relationship in a certain sense with Prob. (P), it is Prob. (1.3), $(1.1)_{2-3}$. We shall prove this relationship to obtain a solution of Prob.(P) via approximate solutions u^n.

This paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, for each $n \in \mathbb{N}$, we prove the local existence and uniqueness of a weak solution u^n of Prob.(P_n). In Section 4, we consider the relationship between Prob.(P_n) and Prob. (P), where we show that the sequence $\{u^n\}$ in appropriate spaces strongly converges to a weak solution u of Prob.(P) as $n \to \infty$. The main tools used here are the linearization method together with the Faedo-Galerkin method and the weak compact method. We end the paper with a remark related to a similar problem.

2 Preliminaries

Put $\Omega = (0,1)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ ess \sup\limits_{0<t<T} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}. \tag{2.1}$$

We put

$$V = \{ v \in H^1(\Omega) : v(1) = 0 \}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + \zeta u(0) v(0), \ u, v \in V. \tag{2.3}$$

V is a closed subspace of H^1 and on V three norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent norms.

We have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2} \|v\|_{H^1} \ \text{for all } v \in H^1. \tag{2.4}$$

Lemma 2.2. *Let $\zeta \ge 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \le \|v_x\| \le \|v\|_a, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \le \|v_x\| \le \|v\|_a \le \sqrt{1+\zeta} \|v\|_{H^1}, \end{cases} \tag{2.5}$$

for all $v \in V$.

Lemma 2.3. *Let $\zeta \ge 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.3) is continuous on $V \times V$ and coercive on V.*

The weak solution of Prob. (1.1) is defined in the following manner.

Definition 2.4. *A function u is called a weak solution of the initial-boundary value problem (1.1) if $u \in V_T = \{ u \in L^\infty(0,T; V \cap H^2) : u' \in L^\infty(0,T;V),\ u'' \in L^\infty(0,T;L^2) \}$ and u satisfies the following variational equation*

$$\langle u''(t), w \rangle + \mu[u](t) a(u(t), w) = \langle f[u](t), w \rangle, \tag{2.6}$$

for all $w \in V$, *a.e.,* $t \in (0, T)$, *together with the initial conditions*

$$u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1, \tag{2.7}$$

where

$$\begin{aligned} f[u](x,t) = &\; f\left(x,t,u(x,t),u_x(x,t),u'(x,t)\right), \\ \mu(t) = &\; \mu[u](t) = 1 + (\bar{S}_n u)(t), \\ (\bar{S}_n u)(t) = &\; \frac{1}{n}\sum_{i=1}^{n} u^2\left(\tfrac{i-1}{n}, t\right). \end{aligned} \tag{2.8}$$

3 Main results

3.1 Existence of a local weak solution

To study the existence and uniqueness of a weak solution of Prob. (1.1), for each $n \in \mathbb{N}$, we make the following assumptions:

$$\begin{aligned} &(H_1) : (\tilde{u}_0, \tilde{u}_1) \in (V \cap H^2) \times V, \;\; \tilde{u}_{0x}(0) - \zeta \tilde{u}_0(0) = 0; \\ &(H_2) : f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^4). \end{aligned}$$

Let $T^* > 0$, for every $T \in (0, T^*]$, we put

$$\begin{aligned} V_T = &\; \{v \in L^\infty(0,T; V \cap H^2) : v' \in L^\infty(0,T;V), \;\; v'' \in L^\infty(0,T;L^2)\}, \\ W_1(T) = &\; C([0,T];V) \cap C^1([0,T];L^2), \end{aligned}$$

it is well known that V_T, $W_1(T)$ are two Banach spaces with respect to the norms (see Lions [8])

$$\begin{aligned} \|v\|_{V_T} = &\; \max\left\{\|v\|_{L^\infty(0,T;V\cap H^2)}, \;\; \|v'\|_{L^\infty(0,T;V)}, \;\; \|v''\|_{L^\infty(0,T;L^2)}\right\}, \\ \|v\|_{W_1(T)} = &\; \|v\|_{C([0,T];V)} + \|v'\|_{C^1([0,T];L^2)}. \end{aligned}$$

For every $M > 0$, we put

$$W(M,T) = \{v \in V_T : \|v\|_{V_T} \le M\}.$$

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W(M,T), \tag{3.1}$$

we associate Prob. (1.1) with the following problem.

Find $u_m \in W(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \mu_m(t) a(u_m(t), w) = \langle F_m(t), w\rangle, \; \forall w \in V, \\ u_m(0) = \tilde{u}_0, \; u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F_m(x,t) = &\; f[u_{m-1}](x,t) = f\left(x,t,u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t)\right), \\ \mu_m(t) = &\; 1 + (\bar{S}_n u_{m-1})(t) = 1 + \frac{1}{n}\sum_{i=1}^{n} u_{m-1}^2\left(\tfrac{i-1}{n}, t\right). \end{aligned} \tag{3.3}$$

Then, we have the following theorem.

Theorem 3.1. *Let $(H_1)-(H_2)$ hold. Then, there are positive constants M and T such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W(M,T)$ defined by (3.1)-(3.3).*

The proof of Theorem 3.1 is based on the linear approximation, the Faedo-Galerkin method (introduced by Lions [8]) together with the techniques of prior estimates. This is similar to the argument in [10] and [11].

Using the results given in Theorem 3.1 and the arguments of compactness, we get the main result in this section as follows

Theorem 3.2. *Let $(H_1)-(H_2)$ hold. Then there exist the constans $M>0$ and $T>0$ chosen as in Theorem 3.1 such that*

(i) Prob. (1.1) has a unique weak solution $u \in W(M,T)$.

(ii) The recurrent sequence $\{u_m\}$ defined by (3.1)-(3.3) converges to the solution u of Prob. (1.1) strongly in $W_1(T)$.

Furthermore, we also have the estimation

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.4}$$

where $0<k_T<1$ and C_T is a positive constant depending only on T, f, $\tilde{u}_0$, $\tilde{u}_1$ and k_T.

Proof. The proof is also similar to the argument in [10] and [11], so we omit the details. □

3.2 Relationship between (P_n) and P

In this section, we consider the following problems

$$(P)\begin{cases} u_{tt} - \left(1+\|u(t)\|^2\right) u_{xx} = f\left(x,t,u,u_x,u_t\right), \ 0<x<1, \ 0<t<T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

and

$$(P_n)\begin{cases} u_{tt} - \left(1+(\bar{S}_n u)(t)\right) u_{xx} = f\left(x,t,u,u_x,u_t\right), \ 0<x<1, \ 0<t<T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x). \end{cases}$$

In what follows, we shall analyze the relationship between (P_n) and (P).

By Theorem 3.2, for each n, (P_n) has a unique weak solution u^n, i.e. u^n satisfies the following variational equation

$$\langle u^n_{tt}(t), w\rangle + \left(1+(\bar{S}_n u)(t)\right) a(u^n(t), w) = \left\langle f\left(\cdot, t, u^n(t), u^n_x(t), u^n_t(t)\right), w\right\rangle, \tag{3.5}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u^n(0) = \tilde{u}_0, \ u^n_t(0) = \tilde{u}_1. \tag{3.6}$$

Moreover, there exist positive constants M, T independing on n such that u^n satisfies

$$u^n \in W(M,T), \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

From (3.7), we deduce that there exists a subsequence of $\{u^n\}$ (still use the same symbol) such that

$$\begin{cases} u^n \to u & \text{in} \quad L^\infty(0,T; V\cap H^2) \text{ weak*}, \\ u^n_t \to u' & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ u^n_{tt} \to u'' & \text{in} \quad L^\infty(0,T;L^2) \text{ weak*}. \end{cases} \tag{3.8}$$

Applying the lemma of Aubin-Lions [8], there exists a subsequence of $\{u^n\}$ (still use the same symbol) such that

$$\begin{cases} u^n \to u & \text{in} \quad C([0,T];V) \text{ strongly}, \\ u_t^n \to u' & \text{in} \quad C([0,T];L^2) \text{ strongly}. \end{cases} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \left|(\bar{S}_n u^n)(t) - (\bar{S}_n u)(t)\right| \leq & \ \frac{1}{n}\sum_{i=1}^{n} \left\|(u^n)^2(t) - u^2(t)\right\|_{C^0(\bar{\Omega})} \\ \leq & \ 2M \left\|u^n(t) - u(t)\right\|_V \\ \leq & \ 2M \left\|u^n(t) - u(t)\right\|_{C([0,T];V)} . \end{aligned} \tag{3.10}$$

By $(3.9)_1$ and (3.10), we obtain

$$\left\|\bar{S}_n u^n - \bar{S}_n u\right\|_{C([0,T]} \leq 2M \left\|u^n(t) - u(t)\right\|_{C([0,T];V)} \to 0. \tag{3.11}$$

Because u^n is the unique weak solution of (P_n), so

$$\begin{aligned} & \int_0^T \langle u_{tt}^n(t), w\rangle \varphi(t)dt + \int_0^T a(u^n(t), w)\varphi(t)dt + \int_0^T (\bar{S}_n u^n)(t) a(u^n(t), w)\varphi(t)dt \\ = & \int_0^T \left\langle f\left(t, u^n(t), u_x^n(t), u_t^n(t)\right) - f\left(t, u(t), u_x(t), u'(t)\right), w\right\rangle \varphi(t)dt \\ & + \int_0^T \left\langle f\left(t, u(t), u_x(t), u'(t)\right), w\right\rangle \varphi(t)dt, \quad \forall w \in V, \forall \varphi \in C_c^\infty(0,T). \end{aligned} \tag{3.12}$$

By $(3.8)_3$ and $(3.9)_1$ we get

$$\begin{aligned} \int_0^T \langle u_{tt}^n(t), w\rangle \varphi(t)dt \ & \to \int_0^T \langle u''(t), w\rangle \varphi(t)dt, \\ \int_0^T a(u^n(t), w)\varphi(t)dt \ & \to \int_0^T a(u(t), w)\varphi(t)dt. \end{aligned} \tag{3.13}$$

We note that

$$\begin{aligned} & \left\|f\left(\cdot, t, u^n(t), u_x^n(t), u_t^n(t)\right) - f\left(\cdot, t, u(t), u_x(t), u'(t)\right)\right\| \\ \leq & \ 2K_M(f)\left[\left\|u^n(t) - u(t)\right\|_V + \left\|u_t^n(t) - u'(t)\right\|\right], \end{aligned}$$

where

$$\begin{aligned} K_M(f) = & \ \|f\|_{C^1\left(\bar{A}_M\right)}, \\ \bar{A}_M = & \ [0,1] \times [0,T^*] \times [-M,M] \times [-\sqrt{2}M, \sqrt{2}M] \times [-M,M]. \end{aligned}$$

Then

$$\begin{aligned} & \left\|f\left(\cdot, t, u^n, u_x^n, u_t^n\right) - f\left(\cdot, t, u, u_x, u'\right)\right\|_{L^2(Q_T)} \\ \leq & \ 2K_M(f)\left[\left\|u^n - u\right\|_{C([0,T];V)} + \left\|u_t^n - u'\right\|_{C([0,T];L^2)}\right] \to 0. \end{aligned} \tag{3.14}$$

Therefore, we conclude that

$$\int_0^T \langle f\left(\cdot,t,u^n(t),u_x^n(t),u_t^n(t)\right) - f\left(\cdot,t,u(t),u_x(t),u'(t)\right),w\rangle \varphi(t)dt \to 0. \tag{3.15}$$

To check that

$$\int_0^T (\bar{S}_n u^n)(t) a(u^n(t),w)\varphi(t)dt \to \int_0^T \|u(t)\|^2 a(u(t),w)\varphi(t)dt, \tag{3.16}$$

we need the lemmas below.

Lemma 3.3. *The following convergence is valid*

$$\left\|\bar{S}_n u - \int_0^1 u^2(y,\cdot)dy\right\|_{L^2(0,T)}^2 = \int_0^T \left|\bar{S}_n u(t) - \int_0^1 u^2(y,t)dy\right|^2 dt \to 0, \ as\ n\to\infty. \tag{3.17}$$

Proof. Note that

$$\frac{1}{n}\sum_{i=1}^n g\left(\tfrac{i-1}{n}\right) \to \int_0^1 g(y)dy, \ \forall g\in C^0\left([0,1]\right). \tag{3.18}$$

Since $u \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;C^0(\bar{\Omega}))$, so the function $y \longmapsto u(y,t)$, a.e. $t\in[0,T]$, belongs to $C^0(\bar{\Omega})$, then

$$(\bar{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^n u^2\left(\tfrac{i-1}{n},t\right) \to \int_0^1 u^2(y,t)dy, \text{ as } n\to\infty. \tag{3.19}$$

On the other hand

$$\left|(\bar{S}_n u)(t)\right| = \frac{1}{n}\sum_{i=1}^n u^2\left(\frac{i-1}{n},t\right) \le \frac{1}{n}\sum_{i=1}^n \|u_x(t)\|^2 \le M^2, \tag{3.20}$$

$$\left|\int_0^1 u^2(y,t)dy\right| \le \|u_x(t)\|^2 \le M^2,$$

so

$$\left|(\bar{S}_n u)(t) - \int_0^1 u^2(y,t)dy\right| \le 2M^2, \text{ for all } n\in\mathbb{N} \text{ and a.e. } t\in[0,T]. \tag{3.21}$$

Applying the dominated convergent theorem, (3.17) is confirmed. Hence, the lemma is proved. □

Lemma 3.4. *The following convergence is valid*

$$\int_0^T (\bar{S}_n u^n)(t) a(u^n(t),w)\varphi(t)dt \to \int_0^T \|u(t)\|^2 a(u(t),w)\varphi(t)dt, \ as\ n\to\infty.$$

Proof. Note that, by (3.12) and Lemma 3.3, it implies that

$$\begin{aligned} &\left\|\bar{S}_n u^n - \int_0^1 u^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\le \left\|\bar{S}_n u^n - \bar{S}_n u\right\|_{L^2(0,T)} + \left\|\bar{S}_n u - \int_0^1 u^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\le 2M\left\|u^n - u\right\|_{L^2(0,T;V)} + \left\|\bar{S}_n u - \int_0^1 u^2(y,\cdot)dy\right\|_{L^2(0,T)} \to 0, \ as\ n\to\infty. \end{aligned} \tag{3.22}$$

Due to (3.9) and (3.22), it leads to

$$\begin{aligned}
&\left|\int_0^T (\bar{S}_n u^n)(t) a(u^n(t), w)\varphi(t)dt - \int_0^T \|u(t)\|^2 a(u(t), w)\varphi(t)dt\right| \qquad (3.23)\\
&\leq \int_0^T \left|\left[(\bar{S}_n u^n)(t) - \|u(t)\|^2\right] a(u^n(t), w)\varphi(t)\right| dt + \\
&\quad \int_0^T \|u(t)\|^2 \left|a(u^n(t) - u(t), w)\varphi(t)\right| dt \\
&\leq \left\|\bar{S}_n u^n - \int_0^1 u^2(y, \cdot)dy\right\|_{L^2(0,T)} \|u^n\|_{L^2(0,T;V)} \|w\|_a \|\varphi\|_{C([0,T])} \\
&\quad + M^2 \|u^n - u\|_{L^2(0,T;V)} \|w\|_a \|\varphi\|_{C([0,T])} \\
&\leq \left[\left\|\bar{S}_n u^n - \int_0^1 u^2(y, \cdot)dy\right\|_{L^2(0,T)} + M\sqrt{T} \|u^n - u\|_{C([0,T];V)}\right] M \|w\|_a \|\varphi\|_{C([0,T])} \\
&\quad \to 0, \text{ as } n \to \infty. \qquad (3.24)
\end{aligned}$$

Hence, the lemma is proved. □

Finally, using Lemmas 3.2 and 3.4 and the convergences (3.14) and (3.16), after letting $n \to \infty$ in (3.13), we conclude that $u \in W(M, T)$ satisfying the equation

$$\begin{aligned}
&\int_0^T \langle u''(t), w\rangle \varphi(t)dt + \int_0^T \left(1 + \|u(t)\|^2\right) a(u(t), w)\varphi(t)dt \qquad (3.25)\\
&= \int_0^T \langle f\left(t, u(t), u_x(t), u'(t)\right), w\rangle \varphi(t)dt,
\end{aligned}$$

for all $w \in V$, $\varphi \in C_c^\infty(0, T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1. \qquad (3.26)$$

Consequently,

$$\begin{cases}
\langle u''(t), w\rangle + \left(1 + \|u(t)\|^2\right) a(u(t), w) \\
\qquad\qquad\qquad = \langle f\left(t, u(t), u_x(t), u'(t)\right), w\rangle, \ \forall w \in V, \\
u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \\
u \in W(M, T).
\end{cases} \qquad (3.27)$$

The uniqueness of a weak solution of (P) is not difficult to prove, let us omit the proof here. Therefore, we have the following theorem.

Theorem 3.5. *Let $(H_1) - (H_2)$ hold. Then there exist positive constants M and T such that*
(i) Prob. (P) has a unique weak solution $u \in W(M, T)$.
(ii) The sequence $\{u^n\}$ converges to the weak solution u of Prob. (P) in the sense

$$\begin{array}{lll}
u^n \to u & \text{in} & L^\infty(0, T; H^2 \cap V) \text{ weak*}, \\
u_t^n \to u' & \text{in} & L^\infty(0, T; V) \text{ weak*}, \\
u_{tt}^n \to u'' & \text{in} & L^\infty(0, T; L^2) \text{ weak*}, \\
u^n \to u & \text{in} & C\left([0, T]; V\right) \cap C^1([0, T]; L^2) \text{ strongly.}
\end{array} \qquad (3.28)$$

3.3 A remark

The methods used in Sections 3, 4 can be applied again to obtain similar results for Prob.(1.1), where $(\bar{S}_n u)(t)$ is defined as follows

$$(\bar{S}_n u)(t) = \frac{1}{n} \sum_{i=0}^{n-1} u^2 \left(\tfrac{i+\theta_i}{n}, t \right),$$

where $\theta_i \in [0,1)$, $i = \overline{0, n-1}$ are given constants.

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