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Asymptotic expansion associated with the Kirchhoff-Carrier-Love equation

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ABSTRACT

In this paper, we consider the Dirichlet boundary problem for a nonlinear wave equation of Kirchhoff-Carrier-Love type as follow

$$\begin{cases} u_{tt} - B\left(\|u(t)\|^2, \|u_x(t)\|^2\right)(u_{xx} + u_{xxtt}) \\ \quad = f(x,t,u,u_x,u_t,u_{xt}) + \sum_{i=1}^{p} \varepsilon_i f_i(x,t,u,u_x,u_t,u_{xt}) \\ \text{for } \; 0 < x < 1, \; 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \; u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1}$$

where $\tilde{u}_0$, $\tilde{u}_1$, B, f, f_i $(i = 1, \cdots, p)$ *are given functions,* $\varepsilon_1, \cdots, \varepsilon_p$ *are small parameters and* $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,dx$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$. *First, a declaration of the existence and uniqueness of solutions provided by the linearly approximate technique and the Faedo-Galerkin method is presented. Then, by using Taylor's expansion for the functions* B, f, f_i, $i = 1, \cdots, p$, *up to* $(N+1)^{th}$ *order, we establish a high-order asymptotic expansion of solutions in the small parameters* $\varepsilon_1, \cdots, \varepsilon_p$.

Keywords: *Kirchhoff-Carrier-Love equation**,** Faedo-Galerkin method; Linear recurrent sequence; Asymptotic expansion.*

1 Introduction

In this paper, we consider the following Dirichlet problem for a Kirchhoff-Carrier-Love equation

$$u_{tt} - B\left(\|u(t)\|^2, \|u_x(t)\|^2\right)(u_{xx} + \lambda u_{xxtt}) = F_{\vec{\varepsilon}}\left(x,t,u,u_x,u_t,u_{xt}\right), \quad \text{for } 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\tilde{u}_0$, $\tilde{u}_1$, B, f, f_i $(i = 1, \cdots, p)$ are given functions and

$$F_{\vec{\varepsilon}}(x,t,u,u_x,u_t,u_{xt}) = f(x,t,u,u_x,u_t,u_{xt}) + \sum_{i=1}^{p} \varepsilon_i f_i(x,t,u,u_x,u_t,u_{xt}),$$

$$\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_p) \in \mathbb{R}^p \text{ and}$$

$$\|u(t)\|^2 = \int_0^1 u^2(x,t)\,dx,$$

$$\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx.$$

In view of its structure, Eq. (1.1) is a very complex model. Apparently, such model equation does not exist in the first place, so we will introduce its development and evolution to show its background by listing several related model equations. We shall show the following model equations not only to illustrate the corresponding physical background, but also to describe the mathematical achievements. When $\Omega = (0, L)$, $B \equiv 1$, $f = f_1 = \cdots = f_p = 0$, Eq. (1.1) is become a Love-type equation as follow

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0. \tag{1.4}$$

Eq. (1.4) was first introduced by V. Radochová [25] to describe the vertical oscillations of a rod, and established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[\frac{1}{2} F\rho \left(u_t^2 + \mu^2 k^2 u_{tx}^2\right) - \frac{1}{2} F \left(E u_x^2 + \rho\mu^2 k^2 u_x u_{xtt}\right)\right] dx, \tag{1.5}$$

where u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, the author obtained a classical solution of Eq. (1.4) associated with the initial conditions (1.3) and the boundary conditions as follow

$$u(0,t) = u(L,t) = 0, \tag{1.6a}$$

or

$$\begin{cases} u(0,t) = 0, \\ \lambda u_{xtt}(L,t) + c^2 u_x(L,t) = 0, \end{cases} \tag{1.6b}$$

where $c^2 = \dfrac{E}{\rho}$, $\lambda = 2\mu^2 k^2$. Further, the asymptotic behaviour of solutions for Prob (1.3), (1.4), (1.6a) (or (1.6b)) as $\lambda \to 0_+$ was also established by the method of small parameters. Before that time, there have been numerous published works of Love-type equations, we refer to some of them as in [3], [7], [17] and references therein.

On the other hand, Love-type equations can be considered as a symmetric version of the regularized long wave equation (a symmetric version of the regularized long wave equation) (SRLW), see [26], was modelled by

$$\begin{cases} u_{xxt} - u_t = \rho_x + u u_x, \\ \rho_t + u_x = 0, \end{cases} \tag{1.7}$$

and describing weakly nonlinear ion acoustic and space - charge waves. Eliminating ρ from (1.7), a class of SRLWE is obtained as follows

$$u_{tt} - u_{xx} - u_{xxtt} = -uu_{xt} - u_x u_t. \tag{1.8}$$

Eq (1.8) is explicitly symmetric in the x and t derivatives, and very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [1] and [2]. The SRLW equations were also arised in a lot of other areas of mathematical physics, see [4], [16] and [23]. It is clear that Eq (1.8) is a special form of Eq. (1.1) in the case $f_i = 0$ for all $i = 1, \cdots, p$ and $f(x, t, u, u_x, u_t, u_{xt}) = -uu_{xt} - u_x u_t$.

A class of well-known equations involved in Eq (1.1) are equations of Kirchhoff type. Indeed, when $\Omega = (0, L)$, $\lambda = 0$, $B = B\left(\|u_x(t)\|^2\right)$ and $F_{\vec{\varepsilon}} = 0$, Eq (1.1) is related to the following equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.9}$$

introduced by Kirchhoff [8], where u is the lateral deflection, L is the length of the string, h is the area of the cross- section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. This equation is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. After its appearance, a lot of of attention is devoted to studying Kirchhoff-type equations. One of early classical studies dedicated to Kirchhoff-type equations was given by Pohozaev [24], and later by Lions [11]. After that, Eq (1.9) has been received a lot of interest in which more abstract models have been proposed, we refer the reader to Cavalcanti et al. [5] and [6], Larkin [9], Medeiros [18]. In addition, the results of mathematical aspects for Kirchhoff model can be found in Medeiros et. al. [19], [20], and the references therein.

In the light of the results mentioned above, the main purpose of this paper is devoted to constructing a high-order asymptotic expansion of solutions in the small parameters $\varepsilon_1, \cdots, \varepsilon_p$ for Prob.(1.1)-(1.3). Meanwhile, in the case $f \in C^1([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$, $B \in C^1(\mathbb{R}^2_+)$, the existence and uniqueness of solutions for the problem provided by the linear approximation and the Faedo-Galerkin method are declared by adopting the similar techniques used in [13], [22], [27] and [28]. The paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we state the existence and uniqueness theorem of solutions for Prob. (1.1) - (1.3). Finally, in Section 4, we establish a high-order asymptotic expansion of the weak solution $u = u(\varepsilon_1, \cdots, \varepsilon_p)$ in the small parameters $\varepsilon_1, \cdots, \varepsilon_p$ for Prob. (1.1) - (1.3) with the requirements $B \in C^{N+1}(\mathbb{R}^2_+)$, $B(y, z) \geq b_* > 0$, for all $(y, z) \in \mathbb{R}^2_+$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $(i = 1, \cdots, p)$. These results can be considered a relative generalization of that given in [12]-[15], [22] and [27].

2 Preliminaries

Put $\Omega = (0,1)$, we use the well-known function spaces denoted by $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notations $\|\cdot\|$ and $\|\cdot\|_X$ respectively stand for the norm in L^2 and the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$ to be Banach space including real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, where

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p dt\right)^{1/p}, & \text{for} \quad 1 \le p < \infty, \\ ess \sup\limits_{0<t<T} ||u(t)||_X, & \text{for} \quad p = \infty. \end{cases}$$

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

We have the following lemma, whose proof is very simple so we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2}\, \|v\|_{H^1} \text{ for all } v \in H^1. \tag{2.2}$$

Remark 2.2. *On H_0^1, $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent norms. Furthermore,*

$$\|v\|_{C^0(\overline{\Omega})} \le \|v_x\| \text{ for all } v \in H_0^1. \tag{2.3}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \triangledown u(t)$, $u_{xx}(t) = \Delta u(t)$ denote by $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f = f(x,t,u,v,w,z)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial v}$, $D_5 f = \frac{\partial f}{\partial w}$, $D_6 f = \frac{\partial f}{\partial z}$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_6^{\alpha_6} f$; $\alpha = (\alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 = N$; $D^{(0,\cdots,0)} f = f$.

Similarly, with $B \in C^N(\mathbb{R}_+^2)$, $B = B(y, z)$, we put $D_1 B = \frac{\partial B}{\partial y}$, $D_2 B = \frac{\partial B}{\partial z}$ and $D^\beta B = D_1^{\beta_1} D_2^{\beta_2} B$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, $|\beta| = \beta_1 + \beta_2 = N$; $D^{(0,0)} B = B$.

Moreover, here Prob. (1.1)-(1.3) will be denoted by $(P_{\vec{\varepsilon}})$, where $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_p)$ and (P_0) respect with $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_p) = (0, \cdots, 0)$.

3 Main results

3.1 The existence and uniqueness theorem

In order to establish the existence and uniqueness theorem, we make the following assumptions:

- (H_1) $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$;
- (H_2) $B \in C^1(\mathbb{R}_+^2)$ and $\exists\, b_* > 0$ such that $B(y,z) \ge b_*$, $\forall (y,z) \in \mathbb{R}_+^2$;
- (H_3) $f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^4)$ and $f(0,t,0,v,0,z) = f(1,t,0,v,0,z) = 0, \forall (t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$.

The weak solution of Prob (1.1)-(1.3) is a function $u \in \tilde{W}_T$, $\tilde{W}_T = \{v \in L^\infty(0,T; H_0^1 \cap H^2) : v', v'' \in L^\infty(0,T; H_0^1 \cap H^2)\}$, such that u satisfies the follwing linear variational problem

$$\langle u''(t), w\rangle + B[u](t) \langle u_x(t) + u_x''(t), w_x\rangle = \langle f[u](t), w\rangle, \tag{3.1}$$

for all $w \in H_0^1$, a.e., $t \in (0,T)$, together with initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{3.2}$$

in which

$$\begin{aligned} B[u](t) &= B\left(\|u(t)\|^2, \|u_x(t)\|^2\right), \\ f[u](x,t) &= f\left(x,t,u(x,t),u_x(x,t),u'(x,t),u'_x(x,t)\right). \end{aligned} \tag{3.3}$$

Consider $T^* > 0$ fixed, for all $T \in (0, T^*]$, we put

$$W_T = \{v \in L^\infty(0,T; H_0^1 \cap H^2) : v_t \in L^\infty(0,T; H_0^1 \cap H^2),\ v_{tt} \in L^\infty(0,T; H_0^1)\} \tag{3.4}$$

is a Banach space with respect to the norm (see Lions [10])

$$\|v\|_{W_T} = \max\{\|v\|_{L^\infty(0,T;H_0^1\cap H^2)},\ \|v'\|_{L^\infty(0,T;H_0^1\cap H^2)},\ \|v''\|_{L^\infty(0,T;H_0^1)}\}. \tag{3.5}$$

For all $M > 0$, we put

$$W_1(M,T) = \{v \in W_T : \|v\|_{W_T} \le M \text{ and } v'' \in L^\infty(0,T; H_0^1 \cap H^2)\}. \tag{3.6}$$

Then we have the following theorem.

Theorem 3.1. *Let $(H_1) - (H_3)$ hold. Then, there exist positive constants M and T such that the problem (P_0) has a unique weak solution $u_0 \in W_1(M,T)$.*

Proof. The proof of Theorem 3.1 is based on the Faedo-Galerkin approximation method (see Lions [10]) together with some similar estimates in [27] and [28]. □

3.2 Asymptotic expansion of solutions in small parameters

In this section, we suppose that the assumptions $(H_1) - (H_3)$ are hold. Then, in order to establish an asymptotic expansion of solutions in small parameters for Prob. (1.1)-(1.3), we need an additional assumption as follow

(H_4) $f_i \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, and $f_i(0,t,0,v,0,z) = f_i(1,t,0,v,0,z) = 0$, $\forall (t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$, $(i = 1, \cdots, p)$.

Consider $T^* > 0$ fixed and let $M > 0$, we put

$$\tilde{K}_M(B) = \|B\|_{C^1([0,M^2]\times[0,M^2])},\ K_M(f) = \|f\|_{C^1(A_M)},$$

where $A_M = \{(x,t,u,v,w,z) \in [0,1] \times [0,T^*] \times \mathbb{R}^4 : |u|, |v|, |w|, |z| \le M\}$.

We consider the problem $(P_{\vec{\varepsilon}})$ depending on p small parameters $\varepsilon_1, \cdots, \varepsilon_p$, with $|\varepsilon_i| < 1$, $i = 1, \cdots, p$:

$$(P_{\vec{\varepsilon}}) \begin{cases} u_{tt} - B(\|u\|^2, \|u_x\|^2)Au = F_{\vec{\varepsilon}}(x,t,u,u_x,u_t,u_{xt}),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \\ Au = u_{xx} + u_{xxtt}, \\ F_{\vec{\varepsilon}}(x,t,u,u_x,u_t,u_{xt}) = f(x,t,u,u_x,u_t,u_{xt}) + \displaystyle\sum_{i=1}^p \varepsilon_i f_i(x,t,u,u_x,u_t,u_{xt}). \end{cases}$$

Under the assumptions $(H_1)-(H_4)$ and by the results of Theorem 3.1, the problem $(P_{\vec{\varepsilon}})$ has a unique weak solution u depending on $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_p)$, namely $u_{\vec{\varepsilon}}=u(\varepsilon_1,\cdots,\varepsilon_p)$. Furthermore, by the fact that $|\varepsilon_i|<1$, $i=1,\cdots,p$, then the solution $u_{\vec{\varepsilon}}$ satisfies

$$u_{\vec{\varepsilon}}\in W_1(M,T),\ \text{for all}\ \vec{\varepsilon},\ \ \|\vec{\varepsilon}\|<1,$$

where positive constants M, T independent on $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_p)$ are similarly chosen as in Theorem 3.1.

Next, we shall study asymptotic expansion of the solution of $(P_{\vec{\varepsilon}})$ with respect to the small parameters $\varepsilon_1,\cdots,\varepsilon_p$.

We use the following notations. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_p)\in\mathbb{Z}_+^p$, and $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_p)\in\mathbb{R}^p$, we put

$$\begin{cases} |\alpha|=\alpha_1+\cdots+\alpha_p,\ \alpha!=\alpha_1!\cdots!\alpha_p!,\\ \|\vec{\varepsilon}\|=\sqrt{\varepsilon_1^2+\cdots+\varepsilon_p^2},\ \vec{\varepsilon}^{\alpha}=\varepsilon_1^{\alpha_1}\cdots\varepsilon_p^{\alpha_p},\\ \alpha,\ \beta\in\mathbb{Z}_+^p,\ \alpha\leq\beta\Longleftrightarrow\alpha_i\leq\beta_i\quad\forall i=1,\cdots,p. \end{cases}\tag{3.7}$$

Then, we have the following lemma.

Lemma 3.2. *Let m, $N\in\mathbb{N}$ and $u_\alpha\in\mathbb{R}$, $\alpha\in\mathbb{Z}_+^p$, $1\leq|\alpha|\leq N$. Then*

$$\left(\sum_{1\leq|\alpha|\leq N}u_\alpha\vec{\varepsilon}^{\alpha}\right)^m=\sum_{m\leq|\alpha|\leq mN}T_N^{(m)}[u]_\alpha\vec{\varepsilon}^{\alpha},\tag{3.8}$$

where the coefficients $T_N^{(m)}[u]_\alpha$, $m\leq|\alpha|\leq mN$ depending on $u=(u_\alpha)$, $\alpha\in\mathbb{Z}_+^p$, $1\leq|\alpha|\leq N$ defined by the recurrence formulas

$$\begin{cases} T_N^{(1)}[u]_\alpha=u_\alpha,\ 1\leq|\alpha|\leq N,\\ T_N^{(m)}[u]_\alpha=\displaystyle\sum_{\beta\in A_\alpha^{(m)}(N)}u_{\alpha-\beta}T_N^{(m-1)}[u]_\beta,\ m\leq|\alpha|\leq mN,\ m\geq 2,\\ A_\alpha^{(m)}(N)=\{\beta\in\mathbb{Z}_+^p:\beta\leq\alpha,\ 1\leq|\alpha-\beta|\leq N,\ m-1\leq|\beta|\leq(m-1)N\}. \end{cases}\tag{3.9}$$

The proof of Lemma 3.2 can be found in [15].

Now, we assume that

(H_5) $\quad B\in C^{N+1}(\mathbb{R}_+^2)$,
$\quad B(y,z)\geq b_*>0$, for all $(y,z)\in\mathbb{R}_+^2$, $(i=1,\cdots,p)$,

(H_6) $\quad f\in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$, $f_i\in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$,
$\quad$ and $f(0,t,0,v,0,z)=f(1,t,0,v,0,z)=f_i(0,t,0,v,0,z)=f_i(1,t,0,v,0,z)=0$,
$\quad$ for all $(t,v,z)\in\mathbb{R}_+\times\mathbb{R}^2$, $(i=1,\cdots,p)$.

Note that u_0 is a unique weak solution of (P_0) (as in Theorem 3.1) defined by

$$(P_0)\begin{cases} u_0''-B[u_0]Au_0=f[u_0],\ 0<x<1,\ 0<t<T,\\ u_0(0,t)=u_0(1,t)=0,\\ u_0(x,0)=\tilde{u}_0(x),\ u_0'(x,0)=\tilde{u}_1(x),\\ u_0\in W_1(M,T). \end{cases}$$

Considering the sequence of weak solutions u_ν, $\nu \in \mathbb{Z}_+^p$, $1 \le |\nu| \le N$, of the following problems

$$(\tilde{P}_\nu) \begin{cases} u_\nu'' - B[u_0]Au_\nu = F_\nu, \ 0 < x < 1, \ 0 < t < T, \\ u_\nu(0,t) = u_\nu(1,t) = 0, \\ u_\nu(x,0) = u_\nu'(x,0) = 0, \\ u_\nu \in W_1(M,T), \end{cases}$$

where F_ν, $\nu \in \mathbb{Z}_+^p$, $1 \le |\nu| \le N$, are defined by the recurrence formulas

$$F_\nu = \begin{cases} f[u_0] \equiv f(x,t,u_0,\nabla u_0,u_0',\nabla u_0'), & |\nu| = 0, \\ \pi_\nu[f] + \sum_{i=1}^{p} \pi_\nu^{(i)}[f_i] + \sum_{\substack{1\le|\alpha|\le N, \\ |\nu-\alpha|\le N}} \rho_\alpha[B]Au_{\nu-\alpha}, & 1 \le |\nu| \le N, \end{cases} \tag{3.10}$$

and $\rho_\nu[B] = \rho_\nu[B;\sigma^{(1)},\sigma^{(2)}]$, $\pi_\nu[f] = \pi_\nu[f;\{u_\gamma\}_{\gamma\le\nu}]$, $\pi_\nu^{(i)}[f] = \pi_\nu^{(i)}[f;\{u_\gamma\}_{\gamma\le\nu}]$, $|\nu| \le N$, are defined as follow.

A/ The fomula $\rho_\nu[B] = \rho_\nu[B,\sigma^{(1)},\sigma^{(2)}]$:

$$\rho_\nu[B] = \rho_\nu[B,\sigma^{(1)},\sigma^{(2)}] \tag{3.11}$$
$$= \begin{cases} B[u_0], & |\nu| = 0, \\ \sum_{|\gamma|\le|\nu|} \frac{1}{\gamma!} D^\gamma B[u_0] \sum_{\substack{\gamma_1\le|\alpha|\le 2\gamma_1 N, \\ \gamma_2\le|\nu-\alpha|\le 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha}, & 1 \le |\nu| \le N, \end{cases}$$

where $\sigma^{(1)} = \left(\sigma_\alpha^{(1)}\right)$, $\sigma^{(2)} = \left(\sigma_\alpha^{(2)}\right)$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le 2N$, are defined by

$$\sigma_\alpha^{(1)} = \begin{cases} 2\langle u_0, u_\alpha\rangle, & |\alpha| = 1, \\ 2\langle u_0, u_\alpha\rangle + \sum_{\beta\le\alpha} \langle u_\beta, u_{\alpha-\beta}\rangle, & 2 \le |\alpha| \le N, \\ \sum_{\beta\le\alpha} \langle u_\beta, u_{\alpha-\beta}\rangle, & N+1 \le |\alpha| \le 2N, \end{cases} \tag{3.12}$$
$$\sigma_\alpha^{(2)} = \begin{cases} 2\langle \nabla u_0, \nabla u_\alpha\rangle, & |\alpha| = 1, \\ 2\langle \nabla u_0, \nabla u_\alpha\rangle + \sum_{\beta\le\alpha} \langle \nabla u_\beta, \nabla u_{\alpha-\beta}\rangle, & 2 \le |\alpha| \le N, \\ \sum_{\beta\le\alpha} \langle \nabla u_\beta, \nabla u_{\alpha-\beta}\rangle, & N+1 \le |\alpha| \le 2N, \end{cases}$$

B/ The fomula $\pi_\nu[f] = \pi_\nu[f;\{u_\gamma\}_{\gamma\le\nu}]$:

$$\pi_\nu[f] = \begin{cases} f[u_0], & |\nu| = 0, \\ \sum_{\substack{1\le|m|\le|\nu| \\ m=(m_1,\cdots,m_4)\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{\substack{(\alpha,\beta,\gamma,\delta)\in A(m,N) \\ \alpha+\beta+\gamma+\delta=\nu}} T_N^{(m_1)}[u]_\alpha & \\ \qquad \times T_N^{(m_2)}[\nabla u]_\beta T_N^{(m_3)}[u']_\gamma T_N^{(m_4)}[\nabla u']_\delta, & 1 \le |\nu| \le N, \end{cases} \tag{3.13}$$

where $m = (m_1, \cdots, m_4) \in \mathbb{Z}_+^4$, $|m| = m_1 + \cdots + m_4$, $m! = m_1! \cdots ! m_4!$, $D^m f = D_3^{m_1} D_4^{m_2} D_5^{m_3} D_6^{m_4} f$, $A(m, N) = \{(\alpha, \beta, \gamma, \delta) \in (\mathbb{Z}_+^p)^4 : m_1 \leq |\alpha| \leq m_1 N,\ m_2 \leq |\beta| \leq m_2 N,\ m_3 \leq |\gamma| \leq m_3 N,\ m_4 \leq |\delta| \leq m_4 N\}$,

$$\begin{cases} \pi_\nu^{(i)}[f] = \pi_{\nu^{(i-)}}[f] = \pi_{\nu_1, \cdots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \cdots, \nu_p}[f],\ i = 1, \cdots, p, \\ \pi_\nu^{(i)}[f] = \pi_{\nu_1, \cdots, \nu_{i-1}, -1, \nu_{i+1}, \cdots, \nu_p}[f] = 0, \text{ if } \nu_i = 0, \\ \nu = (\nu_1, \cdots, \nu_p) \in \mathbb{Z}_+^p,\ \nu^{(i-)} = (\nu_1, \cdots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \cdots, \nu_p). \end{cases} \tag{3.14}$$

Then, we have the following lemma.

Lemma 3.3. *Let $\rho_\nu[B] = \rho_\nu[B, \sigma^{(1)}, \sigma^{(2)}]$, $\pi_\nu[f]$, $|\nu| \leq N$, be the functions defined by the formulas (3.11) and (3.13). Put $h = \sum_{|\gamma| \leq N} u_\gamma \vec{\varepsilon}^\gamma$, then we have*

$$(i)\ B[h] = \sum_{|\nu| \leq N} \rho_\nu[B] \vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^{N+1} \widetilde{R}_N^{(1)}[B, \vec{\varepsilon}], \tag{3.15}$$

$$(ii)\ f[h] = \sum_{|\nu| \leq N} \pi_\nu[f] \vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^{N+1} \bar{R}_N^{(1)}[f, \vec{\varepsilon}], \tag{3.16}$$

with $\left\| \widetilde{R}_N^{(1)}[B, \vec{\varepsilon}] \right\|_{L^\infty(0,T)} + \left\| \bar{R}_N^{(1)}[f, \vec{\varepsilon}] \right\|_{L^\infty(0,T;L^2)} \leq C$, where C is a constant depending only on N, T, f, B, u_γ, $|\gamma| \leq N$.

Proof. (i) In the case of $N = 1$, the proof of (3.15) is easy, hence we omit the details. We only prove it with $N \geq 2$. Put $h = u_0 + \sum_{1 \leq |\alpha| \leq N} u_\alpha \vec{\varepsilon}^\alpha \equiv u_0 + h_1$, we rewrite $B[h]$ as below

$$B[h] = B(\|u_0 + h_1\|^2, \|\nabla u_0 + \nabla h_1\|^2) = B(\|u_0\|^2 + \xi_1, \|\nabla u_0\|^2 + \xi_2), \tag{3.17}$$

where $\xi_1 = \|u_0 + h_1\|^2 - \|u_0\|^2$, $\xi_2 = \|\nabla u_0 + \nabla h_1\|^2 - \|\nabla u_0\|^2$.

By using Taylor's expansion of the function $B(\|u_0\|^2 + \xi_1, \|\nabla u_0\|^2 + \xi_2)$ around the point $(\|u_0\|^2, \|\nabla u_0\|^2)$ up to order $N + 1$, we obtain

$$\begin{aligned} B[h] &= B(\|u_0\|^2 + \xi_1, \|\nabla u_0\|^2 + \xi_2) \\ &= B(\|u_0\|^2, \|\nabla u_0\|^2) + \sum_{1 \leq |\gamma| \leq N} \frac{1}{\gamma!} D^\gamma B(\|u_0\|^2, \|\nabla u_0\|^2) \xi_1^{\gamma_1} \xi_2^{\gamma_2} + R_N[B, u_0, \xi_1, \xi_2] \\ &= B[u_0] + \sum_{1 \leq |\gamma| \leq N} \frac{1}{\gamma!} D^\gamma B[u_0] \xi_1^{\gamma_1} \xi_2^{\gamma_2} + R_N[B, u_0, \xi_1, \xi_2], \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} R_N[B, u_0, \xi_1, \xi_2] &= \sum_{|\gamma| = N+1} \frac{N+1}{\gamma!} \int_0^1 (1-\theta)^N D^\gamma B(\|u_0\|^2 + \theta\xi_1, \|\nabla u_0\|^2 + \theta\xi_2) \xi_1^{\gamma_1} \xi_2^{\gamma_2} d\theta \\ &\equiv \|\vec{\varepsilon}\|^{N+1} R_N^{(1)}[B, u_0, \xi_1, \xi_2]. \end{aligned} \tag{3.19}$$

On the other hand, we have

$$\xi_1 = \|u_0 + h_1\|^2 - \|u_0\|^2 = 2\langle u_0, h_1 \rangle + \|h_1\|^2 \equiv \sum_{1 \leq |\alpha| \leq 2N} \sigma_\alpha^{(1)} \vec{\varepsilon}^\alpha, \tag{3.20}$$

with $\sigma_\alpha^{(1)}$, $1 \leq |\alpha| \leq 2N$ are defined by $(3.12)_1$.

By the formula (3.8), it follows from (3.20) that

$$\xi_1^{\gamma_1} = \left(\sum_{1\leq|\alpha|\leq 2N} \sigma_\alpha^{(1)} \vec{\varepsilon}^\alpha \right)^{\gamma_1} = \sum_{\gamma_1\leq|\alpha|\leq 2\gamma_1 N} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha \vec{\varepsilon}^\alpha, \tag{3.21}$$

where $\sigma^{(1)} = (\sigma_\alpha^{(1)})$, $\alpha \in \mathbb{Z}_+^p$, $1 \leq |\alpha| \leq 2N$.

Similarly, we have

$$\xi_2^{\gamma_2} = \left(\sum_{1\leq|\alpha|\leq 2N} \sigma_\alpha^{(2)} \vec{\varepsilon}^\alpha \right)^{\gamma_2} = \sum_{\gamma_2\leq|\alpha|\leq 2\gamma_2 N} T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_\alpha \vec{\varepsilon}^\alpha, \tag{3.22}$$

where $\sigma^{(2)} = (\sigma_\alpha^{(2)})$, $\alpha \in \mathbb{Z}_+^p$, $1 \leq |\alpha| \leq 2N$, is defined by $(3.12)_2$.

Therefore, it follows from (3.21) and (3.22) that

$$\begin{aligned} \xi_1^{\gamma_1}\xi_2^{\gamma_2} &= \sum_{|\gamma|\leq|\nu|\leq 2|\gamma|N} \left(\sum_{\substack{\gamma_1\leq|\alpha|\leq 2\gamma_1 N,\\ \gamma_2\leq|\nu-\alpha|\leq 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha} \right) \vec{\varepsilon}^\nu \\ &= \sum_{|\gamma|\leq|\nu|\leq 2|\gamma|N} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha]\vec{\varepsilon}^\nu = \\ & \sum_{|\gamma|\leq|\nu|\leq N} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha]\vec{\varepsilon}^\nu + \sum_{N+1\leq|\nu|\leq 2|\gamma|N} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha]\vec{\varepsilon}^\nu \\ &= \sum_{|\gamma|\leq|\nu|\leq N} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha]\vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^{N+1} R_N[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha, \vec{\varepsilon}], \end{aligned} \tag{3.23}$$

where

$$\begin{cases} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha] = \displaystyle\sum_{\substack{\gamma_1\leq|\alpha|\leq 2\gamma_1 N,\\ \gamma_2\leq|\nu-\alpha|\leq 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha}, \\ \|\vec{\varepsilon}\|^{N+1} R_N[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha, \vec{\varepsilon}] = \displaystyle\sum_{N+1\leq|\nu|\leq 2|\gamma|N} \Phi_\nu[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha]\vec{\varepsilon}^\nu. \end{cases} \tag{3.24}$$

Hence, we deduce from (3.18), (3.23) and (3.24) that

$$B[h] = \sum_{|\nu|\leq N} \rho_\nu[B, \sigma^{(1)}, \sigma^{(2)}]\vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^{N+1} \widehat{R}_N^{(1)}[B, u_0, \sigma^{(1)}, \sigma^{(2)}, \xi_1, \xi_2], \tag{3.25}$$

where $\rho_\nu[B] = \rho_\nu[B; \sigma^{(1)}, \sigma^{(2)}]$, $\nu \in \mathbb{Z}_+^p$, $|\nu| \leq N$, is defined by (3.11) and

$$\begin{aligned} \widehat{R}_N^{(1)}[B, u_0, \sigma^{(1)}, \sigma^{(2)}, \xi_1, \xi_2] &= \sum_{1\leq|\gamma|\leq N} \frac{1}{\gamma!} D^\gamma B[u_0] R_N[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \alpha, \vec{\varepsilon}] \\ & + R_N^{(1)}[B, u_0, \xi_1, \xi_2]. \end{aligned} \tag{3.26}$$

By the boundedness of the functions u_γ, u'_γ, $|\gamma| \le N$ in the function space $L^\infty(0,T;H_0^1\cap H^2)$, we obtain from (3.19), (3.24) and (3.26) that $\left\| \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] \right\|_{L^\infty(0,T)} \le C$, where C is a constant only depending on N, T, B, $\|u_\gamma\|_{L^\infty(0,T;L^2)}$, $\|\nabla u_\gamma\|_{L^\infty(0,T;L^2)}$, $|\gamma| \le N$. Hence, the formula (i) of Lemma 3.4 is proved.

(ii) We only prove (3.16) with $N \ge 2$. By using Taylor's expansion of the function $f[u_0+h_1]$ around the point u_0 up to order $N+1$, we obtain from (3.8) that

$$\begin{aligned} f[u_0+h_1] &= f[u_0] + D_3f[u_0]h_1 + D_4f[u_0]\nabla h_1 + D_5f[u_0]h_1' + D_6f[u_0]\nabla h_1' \\ &\quad + \sum_{\substack{2\le|m|\le N \\ m=(m_1,\cdots,m_4)\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} (\nabla h_1')^{m_4} + R_N^{(1)}[f,h_1] \\ &= f[u_0] + D_3f[u_0]h_1 + D_4f[u_0]\nabla h_1 + D_5f[u_0]h_1' + D_6f[u_0]\nabla h_1' \\ &\quad + \sum_{\substack{2\le|m|\le N \\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{|m|\le|\nu|\le N} \tilde{\Phi}_\nu[m,N,f,u,\nabla u,u',\nabla u']\vec{\varepsilon}^\nu \\ &\quad + \sum_{\substack{2\le|m|\le N \\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{N+1\le|\nu|\le|m|N} \tilde{\Phi}_\nu[m,N,f,u,\nabla u,u',\nabla u']\vec{\varepsilon}^\nu + \\ &\quad R_N^{(1)}[f,h_1], \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} &R_N^{(1)}[f,h_1] = \\ &= \sum_{\substack{|m|=N+1 \\ m=(m_1,\cdots,m_4)\in\mathbb{Z}_+^4}} \frac{N+1}{m!} \int_0^1 (1-\theta)^N D^m f[u_0+\theta h_1] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} (\nabla h_1')^{m_4}\, d\theta, \end{aligned}$$

$$\begin{aligned} &\tilde{\Phi}_\nu[m,N,f,u,\nabla u,u',\nabla u'] \\ &= \sum_{\substack{(\alpha,\beta,\gamma,\delta)\in A(m,N) \\ \alpha+\beta+\gamma+\delta=\nu}} T_N^{(m_1)}[u]_\alpha T_N^{(m_2)}[\nabla u]_\beta T_N^{(m_3)}[u']_\gamma T_N^{(m_4)}[\nabla u']_\delta,\ |m| \le |\nu| \le |m|\, N. \end{aligned} \tag{3.28}$$

Note that

$$\begin{aligned} &f[u_0] + D_3f[u_0]h_1 + D_4f[u_0]\nabla h_1 + D_5f[u_0]h_1' + D_6f[u_0]\nabla h_1' \\ &\quad + \sum_{\substack{2\le|m|\le N \\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{|m|\le|\nu|\le N} \tilde{\Phi}_\nu[m,N,f,u,\nabla u,u',\nabla u']\vec{\varepsilon}^\nu \\ &= \sum_{|\nu|\le N} \pi_\nu[f]\vec{\varepsilon}^\nu, \end{aligned} \tag{3.29}$$

where $\pi_\nu[f]$, $1 \le |\nu| \le N$ is defined by (3.13).

Similarly,

$$\begin{aligned} &\sum_{\substack{2\le|m|\le N \\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{N+1\le|\nu|\le|m|N} \tilde{\Phi}_\nu[m,N,f,u,\nabla u,u',\nabla u']\vec{\varepsilon}^\nu + R_N^{(1)}[f,h_1] \\ &\qquad\qquad = \|\vec{\varepsilon}\|^{N+1} \bar{R}_N^{(1)}[f,\vec{\varepsilon}], \end{aligned} \tag{3.30}$$

with $\left\| \bar{R}_N^{(1)}[f,\vec{\varepsilon}] \right\|_{L^\infty(0,T;L^2)} \le C$, C is a constant only depending on N, T, f, u_γ, $|\gamma| \le N$. Then (3.16) hold. Lemma 3.3 is proved. □

Remark 3.4. *Lemma 3.4 is a generalization of a formula given in [13] (p.262, formula (4.38)) and it is useful to obtain Lemma 3.5 below. Lemmas 3.4 and 3.5 are the keys to establish the $(N+1)^{th}$-order asymptotic expansion of the weak solution $u = u(\varepsilon_1, \cdots, \varepsilon_p)$ in the small parameters $\varepsilon_1, \cdots, \varepsilon_p$, which will be presented below.*

Let $u_{\vec{\varepsilon}} = u(\varepsilon_1, \cdots, \varepsilon_p) \in W_1(M,T)$ be a unique weak solution of the problem $(P_{\vec{\varepsilon}})$. Then $v = u_{\vec{\varepsilon}} - \sum_{|\gamma|\le N} u_\gamma \vec{\varepsilon}^\gamma \equiv u_{\vec{\varepsilon}} - h$ satisfies the problem

$$\begin{cases} v'' - B[v+h]Av = F_{\vec{\varepsilon}}[v+h] - F_{\vec{\varepsilon}}[h] + (B[v+h] - B[h])\, Ah \\ \qquad\qquad\qquad\qquad\qquad + E_{\vec{\varepsilon}}(x,t),\ 0<x<1,\ 0<t<T, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \\ Av = \Delta v + \Delta v'', \\ B[v] = B(\|v\|^2, \|v_x\|^2), \\ F_{\vec{\varepsilon}}[v] = f[v] + \sum_{i=1}^p \varepsilon_i f_i[v] = f(x,t,v,v_x,v',v'_x) + \sum_{i=1}^p \varepsilon_i f_i(x,t,v,v_x,v',v'_x), \end{cases} \tag{3.31}$$

where

$$E_{\vec{\varepsilon}}(x,t) = f[h] - f[u_0] + \sum_{i=1}^p \varepsilon_i f_i[h] + (B[h] - B[u_0])\, Ah - \sum_{1\le|\nu|\le N} F_\nu \vec{\varepsilon}^\nu. \tag{3.32}$$

Then, we have the following lemma

Lemma 3.5. *Let (H_1), (H_5) and (H_6) hold. Then there exists a constant $\bar{C}_*$ such that*

$$\|E_{\vec{\varepsilon}}\|_{L^\infty(0,T;L^2)} \le \bar{C}_* \|\vec{\varepsilon}\|^{N+1}, \tag{3.33}$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_i, B, u_γ, $|\gamma| \le N$, $1 \le i \le p$.

Proof. In the case of $N=1$, the proof of Lemma 3.5 is easy, hence we omit the details. We only consider the case $N \ge 2$.

By using the formulas (3.15) and (3.16) for the functions $B[h]$ and $f_i[h]$, we obtain

$$\begin{cases} B[h] = \sum_{|\nu|\le N-1} \rho_\nu[B]\vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[B,\vec{\varepsilon}], \\ f_i[h] = \sum_{|\nu|\le N-1} \pi_\nu[f_i]\vec{\varepsilon}^\nu + \|\vec{\varepsilon}\|^N \bar{R}_{N-1}^{(1)}[f_i,\vec{\varepsilon}],\ 1\le i\le p. \end{cases} \tag{3.34}$$

By (3.14) and $(3.34)_2$, we rewrite $\varepsilon_i f_i[h]$, $1 \le i \le p$, as follows

$$\begin{aligned} \varepsilon_i f_i[h] &= \sum_{|\nu|\le N-1} \pi_\nu[f_i]\varepsilon_i\vec{\varepsilon}^\nu + \varepsilon_i \|\vec{\varepsilon}\|^N \bar{R}_{N-1}^{(1)}[f_i,\vec{\varepsilon}] \\ &= \sum_{1\le|\nu|\le N,\ \nu_i\ge1} \pi_{\nu_1,\nu_2,\cdots,\nu_{i-1},\nu_i-1,\nu_{i+1},\cdots,\nu_p}[f_i]\vec{\varepsilon}^\nu + \varepsilon_i \|\vec{\varepsilon}\|^N \bar{R}_{N-1}^{(1)}[f_i,\vec{\varepsilon}] \\ &= \sum_{1\le|\nu|\le N} \pi_\nu^{(i)}[f_i]\vec{\varepsilon}^\nu + \varepsilon_i \|\vec{\varepsilon}\|^N \bar{R}_{N-1}^{(1)}[f_i,\vec{\varepsilon}]. \end{aligned} \tag{3.35}$$

we deduce from (3.16) and (3.35) that

$$
\begin{aligned}
& f[h] - f[u_0] + \sum_{i=1}^{p} \varepsilon_i f_i[h] \\
&= \sum_{1\le|\nu|\le N} \left[\pi_\nu[f] + \sum_{i=1}^{p} \pi_\nu^{(i)}[f_i] \right] \vec{\varepsilon}^{\nu} + \|\vec{\varepsilon}\|^{N+1} \bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \vec{\varepsilon}],
\end{aligned} \tag{3.36}
$$

where $\bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \vec{\varepsilon}] = \bar{R}_N^{(1)}[f, \vec{\varepsilon}] + \sum_{i=1}^{p} \frac{\varepsilon_i}{\|\vec{\varepsilon}\|} \bar{R}_{N-1}^{(1)}[f_i, \vec{\varepsilon}]$ is bounded in $L^\infty(0,T;L^2)$ by a constant only depending on N, T, f, f_i, u_γ, $|\gamma| \le N$, $1 \le i \le p$.

On the other hand, we deduce from (3.15) that

$$
\begin{aligned}
& \left(B[h] - B[u_0]\right) Ah \\
&= \sum_{1\le|\nu|\le 2N} \sum_{\substack{1\le|\alpha|\le N,\\ |\nu-\alpha|\le N}} \left(\rho_\alpha[B]\right) A u_{\nu-\alpha} \vec{\varepsilon}^{\nu} + \|\vec{\varepsilon}\|^{N+1} \widetilde{R}_N^{(1)}[B, \vec{\varepsilon}],
\end{aligned} \tag{3.37}
$$

where

$$
\widetilde{R}_N^{(1)}[B, \vec{\varepsilon}] = \widetilde{R}_N^{(1)}[B, \vec{\varepsilon}] Ah. \tag{3.38}
$$

We decompose the sum $\sum_{1\le|\nu|\le 2N}$ into the addition of two sums $\sum_{1\le|\nu|\le N}$ and $\sum_{N+1\le|\nu|\le 2N}$. Hence, we rewritte (3.36) as below

$$
\left(B[h] - B[u_0]\right) Ah = \sum_{1\le|\nu|\le N} \sum_{\substack{1\le|\alpha|\le N,\\ |\nu-\alpha|\le N}} \left(\rho_\alpha[B]\right) A u_{\nu-\alpha} \vec{\varepsilon}^{\nu} + \|\vec{\varepsilon}\|^{N+1} \widetilde{R}_N^{(2)}[B, \vec{\varepsilon}], \tag{3.39}
$$

where

$$
\begin{aligned}
\|\vec{\varepsilon}\|^{N+1} \widetilde{R}_N^{(2)}[B, \vec{\varepsilon}] &= \|\vec{\varepsilon}\|^{N+1} \widetilde{R}_N^{(1)}[B, \vec{\varepsilon}] \\
&\quad + \sum_{N+1\le|\nu|\le 2N} \sum_{\substack{1\le|\alpha|\le N,\\ |\nu-\alpha|\le N}} \left(\rho_\alpha[B]\right) A u_{\nu-\alpha} \vec{\varepsilon}^{\nu}.
\end{aligned} \tag{3.40}
$$

Combining (3.10), (3.11), (3.13), (3.32), (3.36) and (3.39), then we obtain

$$
E_{\vec{\varepsilon}} = \|\vec{\varepsilon}\|^{N+1} \left[\bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \vec{\varepsilon}] + \widetilde{R}_N^{(2)}[B, \vec{\varepsilon}] \right]. \tag{3.41}
$$

By the functions $u_\nu \in W_1(M,T)$, $|\nu| \le N$, we obtain from (3.36) and (3.40) that

$$
\|E_{\vec{\varepsilon}}\|_{L^\infty(0,T;L^2)} = \|\vec{\varepsilon}\|^{N+1} \left\| \bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \vec{\varepsilon}] + \widetilde{R}_N^{(2)}[B, \vec{\varepsilon}] \right\|_{L^\infty(0,T;L^2)} \le \bar{C}_* \|\vec{\varepsilon}\|^{N+1}, \tag{3.42}
$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_i, B, u_γ, $|\gamma| \le N$, $1 \le i \le p$. The proof of Lemma 3.5 is complete. □

Now, we consider the sequence of functions $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ v_m'' - B[v_{m-1}+h]Av_m = F_{\vec{\varepsilon}}[v_{m-1}+h] - F_{\vec{\varepsilon}}[h] + (B[v_{m-1}+h] - B[h])\,Ah \\ \qquad\qquad\qquad\qquad\qquad\qquad + E_{\vec{\varepsilon}}(x,t),\ 0<x<1,\ 0<t<T, \\ v_m(0,t) = v_m(1,t) = 0, \\ v_m(x,0) = v_m'(x,0) = 0,\ m \geq 1. \end{cases} \tag{3.43}$$

With $m = 1$, we have the problem

$$\begin{cases} v_1'' - B[h]Av_1 = E_{\vec{\varepsilon}}(x,t),\ 0<x<1,\ 0<t<T, \\ v_1(0,t) = v_1(1,t) = 0, \\ v_1(x,0) = v_1'(x,0) = 0. \end{cases} \tag{3.44}$$

By multiplying two sides of (3.44) by v_1', we verify without difficulty from (3.33) that

$$\begin{aligned} &||v_1'(t)||^2 + \bar{B}_1(t)\left(\|v_{1x}(t)\|^2 + \|v_{1x}'(t)\|^2\right) \\ &= \int_0^t \bar{B}_1'(s)\left(\|v_{1x}(s)\|^2 + \|v_{1x}'(s)\|^2\right)ds + 2\int_0^t \langle E_{\vec{\varepsilon}}(s), v_1'(s)\rangle ds \\ &\leq T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v_1'(s)\|^2\,ds + \int_0^t \left|\bar{B}_1'(s)\right|\left(\|v_{1x}(s)\|^2 + \|v_{1x}'(s)\|^2\right)ds, \end{aligned} \tag{3.45}$$

where $\bar{B}_1(t) = B[h](t) = B(\|h(t)\|^2, \|\nabla h(t)\|^2)$.

By

$$\bar{B}_1'(t) = 2D_1B[h]\langle h(t), h'(t)\rangle + 2D_2B[h]\langle \nabla h(t), \nabla h'(t)\rangle, \tag{3.46}$$

we have

$$\left|\bar{B}_1'(t)\right| \leq 4M_*^2\tilde{K}_{M_*}(B) \equiv \zeta_1,\ \text{for all}\ \|\overrightarrow{\varepsilon}\| < 1, \tag{3.47}$$

with $M_* = N_1M$, and $N_1 = card\{\gamma \in \mathbb{Z}_+^p : |\gamma| \leq N\}$. It follows from (3.45), (3.47) that

$$\begin{aligned} &||v_1'(t)||^2 + b_*\left(\|v_{1x}(t)\|^2 + \|v_{1x}'(t)\|^2\right) \\ &\leq T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2} + (1+\zeta_1)\int_0^t \left(\|v_{1x}(s)\|^2 + \|v_{1x}'(s)\|^2\right)ds. \end{aligned} \tag{3.48}$$

By Gronwall's lemma, we obtain from (3.48) that

$$\|v_{1x}(t)\|^2 + \|v_{1x}'(t)\|^2 \leq \frac{1}{b_*}T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2}\exp\left[(1+\zeta_1)\,T\right]. \tag{3.49}$$

Hence

$$||v_1||_{C^1([0,T];H_0^1)} \leq \frac{2}{\sqrt{b_*}}\sqrt{T}\bar{C}_*\,\|\vec{\varepsilon}\|^{N+1}\exp\left[\frac{1}{2}(1+\zeta_1)\,T\right]. \tag{3.50}$$

We shall prove that there exists a constant C_T independent of m and $\vec{\varepsilon}$,such that

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq C_T\,\|\vec{\varepsilon}\|^{N+1},\ \text{with}\ \|\vec{\varepsilon}\| < 1,\ \text{for all}\ m. \tag{3.51}$$

By multiplying two sides of (3.43) with v'_m and after integrating in t, we obtain from (3.33) that

$$\begin{aligned} &||v'_m(t)||^2 + \bar{B}_m(t)\left(\|v_{mx}(t)\|^2 + \|v'_{mx}(t)\|^2\right) \\ &\leq T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v'_m(s)\|^2\,ds + \int_0^t \bar{B}'_m(s)\left(\|v_{mx}(s)\|^2 + \|v'_{mx}(s)\|^2\right)ds \\ &\quad + 2\int_0^t \langle F_{\vec{\varepsilon}}[v_{m-1}+h] - F_{\vec{\varepsilon}}[h], v'_m(s)\rangle ds + \\ &\quad 2\int_0^t \left(B[v_{m-1}+h] - B[h]\right)\langle Ah(s), v'_m(s)\rangle ds \\ &\equiv T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v'_m(s)\|^2\,ds + \widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3, \end{aligned} \tag{3.52}$$

with $\bar{B}_m(t) = B[v_{m-1}+h](t) = B(\|v_{m-1}(t)+h(t)\|^2), \|\nabla v_{m-1}(t) + \nabla h(t)\|^2)$.

We now estimate the integrals on the right – hand side of (3.52) as follows.

Estimating $\widehat{J}_1$. We have

$$\begin{aligned} \bar{B}'_m(t) &= 2D_1B[v_{m-1}+h](t)\langle v_{m-1}+h, v'_{m-1}+h'\rangle \\ &\quad + 2D_2B[v_{m-1}+h](t)\langle \nabla v_{m-1}+\nabla h, \nabla v'_{m-1}+\nabla h'\rangle, \end{aligned} \tag{3.53}$$

hence

$$\left|\bar{B}'_m(t)\right| \leq 4\bar{M}_*^2\tilde{K}_{\bar{M}_*}(B) \equiv \bar{\zeta}_1, \text{ for all } \vec{\varepsilon},\ \|\vec{\varepsilon}\| < 1, \tag{3.54}$$

with $\bar{M}_* = (1+N_1)M$.

It follows from (3.54), that

$$\widehat{J}_1 = \int_0^t \bar{B}'_m(s)\left(\|v_{mx}(s)\|^2 + \|v'_{mx}(s)\|^2\right)ds \leq \bar{\zeta}_1\int_0^t \left(\|v_{mx}(s)\|^2 + \|v'_{mx}(s)\|^2\right)ds. \tag{3.55}$$

Estimating $\widehat{J}_2$. Note that $\|f[v_{m-1}+h] - f[h]\| \leq 2K_{\bar{M}_*}(f)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}$, $\|f_i[v_{m-1}+h] - f_1[h]\| \leq 2K_{\bar{M}_*}(f_i)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}$, hence, we have

$$\|F_{\vec{\varepsilon}}[v_{m-1}+h] - F_{\vec{\varepsilon}}[h]\| \leq \bar{\zeta}_2\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}, \tag{3.56}$$

where $\bar{\zeta}_2 = \bar{\zeta}_2(M, f, f_1, \cdots, f_p) = 2K_{\bar{M}_*}(f) + 2\sum_{i=1}^{p} K_{\bar{M}_*}(f_i)$. Therefore, we deduce from (3.56) that

$$\begin{aligned} \widehat{J}_2 &= 2\int_0^t \|F_{\vec{\varepsilon}}[v_{m-1}+h] - F_{\vec{\varepsilon}}[h]\|\,\|v'_m(s)\|\,ds \\ &\leq T\bar{\zeta}_2^2\,\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} + \int_0^t \|v'_m(s)\|^2\,ds. \end{aligned} \tag{3.57}$$

Estimating $\widehat{J}_3$. First, we need an estimation of $|B[v_{m-1}+h] - B[h]|$.

From the inequality

$$|B[v_{m-1}+h] - B[h]| \leq 4\bar{M}_*\tilde{K}_{\bar{M}_*}(B)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)},$$

it follows that

$$|B[v_{m-1}+h]-B[h]| \leq 4\bar{M}_*\tilde{K}_{\bar{M}_*}(B)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}\,. \tag{3.58}$$

We remark that

$$\|Ah(s)\| \leq \sum_{1\leq|\alpha|\leq N}\|Au_\alpha(s)\|\,\|\vec{\varepsilon}^\alpha\| \leq \sum_{1\leq|\alpha|\leq N}\|Au_\alpha(s)\| \leq 2N_1M = 2M_*. \tag{3.59}$$

Hence, we deduce from (3.58) and (3.59) that

$$\begin{aligned}\widehat{J}_3 &= 2\int_0^t (B[v_{m-1}+h]-B[h])\,\langle Ah(s), v_m'(s)\rangle ds \\ &\leq T\bar{\zeta}_3^2\,\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} + \int_0^t \|v_m'(s)\|^2\,ds,\end{aligned} \tag{3.60}$$

in which $\bar{\zeta}_3 = \bar{\zeta}_3(M,B) = 8M_*\bar{M}_*\tilde{K}_{\bar{M}_*}(B)$.
Combining (3.52), (3.55), (3.57) and (3.60), then we obtain

$$\begin{aligned}&\|v_m'(t)\|^2 + \bar{B}_m(t)\left(\|v_{mx}(t)\|^2 + \|v_{mx}'(t)\|^2\right) \\ &\leq T\bar{C}_*^2\,\|\vec{\varepsilon}\|^{2N+2} + T\left(\bar{\zeta}_2^2+\bar{\zeta}_3^2\right)\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} + \\ &\left(3+\bar{\zeta}_1\right)\int_0^t\left(\|v_{mx}(s)\|^2 + \|v_{mx}'(s)\|^2\right)ds.\end{aligned} \tag{3.61}$$

By using Gronwall's lemma, we deduce from (3.61) that

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq \sigma_T\,\|v_{m-1}\|_{C^1([0,T];H_0^1)} + \delta,\ \text{ for all } m\geq 1, \tag{3.62}$$

with $\sigma_T = \eta_T\sqrt{\bar{\zeta}_2^2+\bar{\zeta}_3^2}$, $\delta = \eta_T\bar{C}_*\,\|\vec{\varepsilon}\|^{N+1}$, $\eta_T = \sqrt{\frac{T}{b_*}}\exp\left(\frac{1}{2b_*}T\left(3+\bar{\zeta}_1\right)\right)$.
Assuming that

$$\sigma_T < 1,\ \text{ with a suitable constant } T>0. \tag{3.63}$$

We can easily prove the following lemma.

Lemma 3.6. *Let the sequence* $\{z_m\}$ *satisfy*

$$z_m \leq \sigma z_{m-1} + \delta \text{ for all } m\geq 1,\ z_0 = 0, \tag{3.64}$$

where $0\leq\sigma<1$, $\delta\geq 0$ *are given constants. Then*

$$z_m \leq \delta/(1-\sigma) \text{ for all } m\geq 1.\ \square \tag{3.65}$$

Applying Lemma 3.5 to (3.62) in the case $z_m = \|v_m\|_{C^1([0,T];H_0^1)}$, $\sigma = \sigma_T = \eta_T\sqrt{\bar{\zeta}_2^2+\bar{\zeta}_3^2} < 1$, $\delta = \eta_T\bar{C}_*\,\|\vec{\varepsilon}\|^{N+1}$, *it follows from (3.65) that*

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq \delta/(1-\sigma_T) = C_T\,\|\vec{\varepsilon}\|^{N+1}, \tag{3.66}$$

where $C_T = \dfrac{\eta_T\bar{C}_*}{1-\eta_T\sqrt{\bar{\zeta}_2^2+\bar{\zeta}_3^2}}$.

On the other hand, by using the linear approximation method in [27], the linear recurrent sequence $\{v_m\}$ defined by (3.43) converges strongly in the space $C^1([0,T];H_0^1)$ to the solution v of Prob (3.31). Hence, as $m \to +\infty$ in (3.66), we get that $\|v\|_{C^1([0,T];H_0^1)} \leq C_T \|\vec{\varepsilon}\|^{N+1}$. This implies that

$$\left\| u_{\vec{\varepsilon}} - \sum_{|\gamma| \leq N} u_\gamma \vec{\varepsilon}^\gamma \right\|_{C^1([0,T];H_0^1)} \leq C_T \|\vec{\varepsilon}\|^{N+1}. \tag{3.67}$$

Finally, we summarize the obtained results in the following theorem.

Theorem 3.7. *Let (H_1), (H_5) and (H_6) hold. Then there exist constants $M > 0$ and $T > 0$ such that, for all $\vec{\varepsilon}$, with $\|\vec{\varepsilon}\| < 1$, the problem $(P_{\vec{\varepsilon}})$ has a unique weak solution $u_{\vec{\varepsilon}} \in W_1(M,T)$ satisfying an asymptotic estimation up to order $N+1$ as in (3.67), where the functions u_ν, $|\nu| \leq N$ are the weak solutions of $(\tilde{P}_\nu)$, $|\nu| \leq N$, respectively* $\square$

Remark 3.8. *Typical examples about asymptotic expansion of solutions in a small parameter can be found in some works, see* [12]- [14], [21]. *In the case of many small parameters, there are only few results, for example, see* [15] *and* [22] *respectively to the asymptotic expansion of solutions in two and three small parameters.*

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