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A new class of bilevel weak vector variational inequality problems

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ABSTRACT

In this paper, we first introduce a new class of bilevel weak vector variational inequality problems in locally convex Hausdorff topological vector spaces. Then, using the Kakutani-Fan-Glicksberg fixed-point theorem, we establish some existence conditions of the solution for this problem.

Keywords: *Bilevel weak vector variational inequality problems; Kakutani-Fan-Glicksberg fixed-point theorem; existence conditions*

1 Introduction and Preliminaries

It is known that, the existence conditions of solutions of optimization-related problems is one of the important topics in optimization theory and so many authors have tried to find several good conditions of the existence of solution sets of various problems as optimization problems, complementarity problems traffic network problems, equilibrium problems [5, 8, 9] and the references therein.

On the other hand, Mordukhovich [12] introduced equilibrium problems with equilibrium constraints and studied optimal conditions to this problem in 2004. In recent years, equilibrium problems with equilibrium constraints have been attracted by many authors in different directions, for example, the existence conditions of solutions [3, 6, 10], the stability properties of solutions [6, 7, 2]. However, to the best of our knowledge, up to now, there have not been any works on the existence conditions of solutions of bilevel weak vector variational inequality problems.

Motivated and inspired by the above, in this paper, we investigate the existence conditions of solutions for bilevel weak vector variational inequality problems in locally convex Hausdorff topological vector spaces. Let X, Z be real locally convex Hausdorff topological vector spaces, $L(X, Z)$ be the space of all linear continuous operators from X into Z, A be a nonempty compact subset of X and $C_1 \subset Z$ be a closed convex and pointed cone with $\text{int}C_1 \neq \emptyset$, where $\text{int}C_1$ is the interior of C_1. Let $K : A \rightrightarrows A$ and $T : A \rightrightarrows L(X, Z)$ be multifunctions, $\eta : A \times A \to A$ be a continuous single-valued mapping. Denoted $\langle z, x\rangle$ by the value of a linear operator $z \in L(X; Y)$ at $x \in A$, we always assume that $\langle ., .\rangle : L(X; Z) \times A \to Z$ is continuous. We consider the following *weak vector quasi-variational inequality problems*:

(WQVIP) Find $x \in A$ such that, there exists $z \in T(x)$ satisfying

$$\begin{cases} x \in K(x) \\ \langle z, \eta(y, x)\rangle \in Z \setminus -\text{int}C_1 \text{ for all } y \in K(x). \end{cases}$$

We denote the solution set of the problem (WQVIP) by $\mathbb{Q}(K, T)$.

Let P be a real locally convex Hausdorff topological vector space, $L(X, P)$ be the space of all linear continuous operators from X into P, $C_2 \subset P$ be a closed convex and pointed cone with $\text{int}C_2 \neq \emptyset$ and $H : A \to L(X, P)$ be a single-valued mapping.

Also, we consider the following *weak bilevel vector variational inequality problems*:

(WBVIP) Find a point $x \in \mathbb{Q}(K, T)$ such that

$$\langle H(x), y - x\rangle \in P \setminus -\text{int}C_2, \ \ \forall y \in \mathbb{Q}(K, T);$$

where $\mathbb{Q}(K, T)$ be the solution set of the weak vector quasi-variational inequality problems. We denote the solution set of the problem (WBVIP) by $\mathbb{O}(H)$.

Now, we recall the following well-known definitions and some results for the main results:

Definition 1.1 (see [1]) Let X, Y be two topological vector spaces, $F : X \rightrightarrows Y$ be a multifunction and let $x_0 \in X$ be a given point.

(1) F is said to be *lower semi-continuous* (lsc) at $x_0 \in X$ if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.

(2) F is said to be *upper semi-continuous* (usc) at $x_0 \in X$ if, for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq F(x)$ for all $x \in N$.

(3) F is said to be *continuous* at $x_0 \in X$ if it is both lsc and usc at $x_0 \in X$

(4) F is said to be *closed* at x_0 if, for each of the nets $\{x_\alpha\}$ in X converging to x_0 and $\{y_\alpha\}$ in Y converging to y_0 such that $y_\alpha \in F(x_\alpha)$, we have $y_0 \in F(x_0)$.

If $A \subset X$, then F is said to be *usc* (*lsc*, *continuous*, *closed*, respectively) on the set A if F is usc (lsc, continuous, closed, respectively) at all $x \in \mathrm{dom}F \cap A$. If $A \equiv X$, then we omit "on X" in the statement.

Lemma 1.1 (see [1]) *Let X, Y be two topological vector spaces and $F : X \rightrightarrows Y$ be a multifunction. Then we have the following:*

(1) *If F is upper semi-continuous with closed values, then F is closed.*

(2) *If F is closed and $F(X)$ is compact, then F is upper semi-continuous.*

Lemma 1.2 (see [1]) *Let X, Y be two topological vector spaces and $F : X \rightrightarrows Y$ be a multifunction. Then we have the following:*

(1) *F is lower semi-continuous $x_0 \in X$ if and only if, for each net $\{x_\alpha\} \subseteq X$ which converges to $x_0 \in X$ and for each $y_0 \in F(x_0)$, there exists $\{y_\alpha\}$ in Y such that $y_\alpha \in F(x_\alpha)$, $y_\alpha \to y_0$.*

(2) *If F has compact values, then F is upper semi-continuous $x_0 \in X$ if and only if, for each net $\{x_\alpha\} \subseteq X$ which converges to $x_0 \in X$ and for each net $\{y_\alpha\}$ in Y such that $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 1.3 (see [4]) *Let A be a nonempty convex compact subset of Hausdorff topological vector space X and N be a subset of $A \times A$ such that*

(i) *for each at* $x \in A$, $(x, x) \notin N$;

(ii) *for each at* $y \in A$, *the set* $\{x \in A : (x, y) \in N\}$ *is open on* A;

(iii) *for each at* $x \in A$, *the set* $\{y \in A : (x, y) \in N\}$ *is convex or empty.*

Then there exists $x_0 \in A$ *such that* $(x_0, y) \notin N$ *for all* $y \in A$.

Lemma 1.4 (see [11]) *Let* A *be a nonempty compact convex subset of a locally convex Hausdorff vector topological space* X*. If* $F : A \rightrightarrows A$ *is upper semi-continuous and, for any* $x \in A$, $F(x)$ *is nonempty convex closed, then there exists* $x^* \in A$ *such that* $x^* \in F(x^*)$.

2 Main Results

In this section, we establish some existence results for weak bilevel vector quasi-variational inequality problems.

We first introduce the concept of weakly C-quasiconvexity.

Definition 2.1 Let X, Z be two topological vector spaces, A be a nonempty closed subset of X, and $C \subset Z$ is a solid pointed closed convex cone and $f : A \to Z$ be a function.The mapping f is said to be *weakly* C*-quasiconvex* on $A \subset X$ if, for each $x_1, x_2 \in A$, $\lambda \in [0, 1]$ with $f(x_1) \in Z \setminus -\text{int}C$ and $f(x_2) \in Z \setminus -\text{int}C$, we have

$$f((1-\lambda)x_1 + \lambda x_2) \in Z \setminus -\text{int}C,$$

We now establish some existence conditions of solution sets of the weak vector quasi-variational inequality problems.

Lemma 2.1 *Let* X, Z *be real locally convex Hausdorff topological vector spaces,* $L(X, Z)$ *be the space of all linear continuous operators from* X *into* Z*,* A *be a nonempty compact subset of* X *and* $C_1 \subset Z$ *be a closed convex and pointed cone with* $\text{int}C_1 \neq \emptyset$*, where* $\text{int}C_1$ *is the interior of* C_1*. Let* $K : A \rightrightarrows A$ *and* $T : A \rightrightarrows L(X, Z)$ *be multifunctions,* $\eta : A \times A \to A$ *be a continuous single-valued mapping. Denoted* $\langle z, x \rangle$ *by the value of a linear operator* $z \in L(X; Z)$ *at* $x \in A$*, we always assume that* $\langle ., . \rangle : L(X; Z) \times A \to Z$ *is continuous. Suppose the following conditions:*

(i) K *is continuous on* A *with nonempty compact convex values;*

(ii) *T is upper semicontinuous on A with nonempty compact values;*

(iii) *for all $x \in A, z \in L(X;Z)$, $\langle z, \eta(x,x)\rangle \in Z \setminus -\text{int}C_1$;*

(iv) *for all $x \in A, z \in L(X;Z)$, the set $\{y \in A : \langle z, \eta(y,x)\rangle \notin Z \setminus -\text{int}C_1\}$ is convex;*

(v) *for all $y \in A, z \in L(X;Z)$, the map $x \mapsto \langle z, \eta(y,x)\rangle$ is weakly C_1-quasiconvex, i.e., for all x_1, $x_2 \in A$ and all $\lambda \in [0,1]$, $y \in A, z \in L(X;Z)$, we have*

$$\langle z, \eta(y,x_1)\rangle \in Z \setminus -\text{int}C_1 \text{ and } \langle z, \eta(y,x_2)\rangle \in Z \setminus -\text{int}C_1$$
$$\Longrightarrow \langle z, \eta(y, \lambda x_1 + (1-\lambda)x_2)\rangle \in Z \setminus -\text{int}C_1;$$

(vi) *the set $\{(x,y,z) \in A \times A \times L(X,Z) : \langle z, \eta(y,x)\rangle \in Z \setminus -\text{int}C_1\}$ is closed.*

Then the weak vector quasi-variational inequality problem has a solution, i.e., there exist $\bar{x} \in A$ and $\bar{z} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ satisfying

$$\langle \bar{z}, \eta(y,\bar{x})\rangle \in Z \setminus -\text{int}C_1, \forall y \in K(\bar{x}).$$

Moreover, the solution set of the weak vector quasi-variational inequality problem is compact.

Proof. For all $x \in A, z \in L(X,Z)$, we define a multifunction $\mathbb{M} : A \times L(X,Z) \rightrightarrows A$ by

$$\mathbb{M}(x,z) = \{a \in K(x) : \langle z, \eta(y,a)\rangle \in Z \setminus -\text{int}C_1, \ \forall y \in K(x)\}.$$

First, we show that $\mathbb{M}(x,z)$ is nonempty. Indeed, for every $x \in A$, $K(x)$ is nonempty compact convex set. Set

$$\mathbb{N} = \{(a,y) \in K(x) \times K(x) : \langle z, \eta(y,a)\rangle \notin Z \setminus -\text{int}C_1\}.$$

By the condition (iii), we have for any $a \in K(x)$, $(a,a) \in \mathbb{N}$. It follows from the condition (iv) that the set $\{y \in K(x) : (a,y) \notin \mathbb{N}\}$ is convex. Moreover, by the condition (iv), we have for any $a \in K(x)$, the set $\{y \in K(x) : (a,y) \in \mathbb{N}\}$ is open. So, by Lemma 1.3 there exists $a^* \in K(x)$ such that $(a^*, y) \notin \mathbb{N}$, for all $y \in K(x)$, i.e.,

$$\langle z, \eta(y,a^*)\rangle \in Z \setminus -\text{int}C_1, \forall y \in K(x).$$

Hence, $\mathbb{M}(x,z)$ is nonempty.

Second, we verify that $\mathbb{M}(x,z)$ is a convex set. In fact, let $a_1, a_2 \in \mathbb{M}(x,z)$, $\lambda \in [0,1]$ and put $a = \lambda a_1 + (1-\lambda)a_2$. Since $a_1, a_2 \in K(x)$ and $K(x)$ is a convex set, we have $a \in K(x)$. From $a_1, a_2 \in \mathbb{M}(x,z)$, it follows that, for any $y \in K(x)$, we have

$$\langle z, \eta(y,a_1)\rangle \in Z \setminus -\mathrm{int}C_1 \text{ and } \langle z, \eta(y,a_2)\rangle \in Z \setminus -\mathrm{int}C_1.$$

By the condition (v), since the map $x \mapsto \langle z, \eta(y,x)\rangle$ is weakly C_1-quasiconvex, we have

$$\langle z, \eta(y, \lambda x_1 + (1-\lambda)x_2)\rangle \in Z \setminus -\mathrm{int}C_1, \ \ \forall \lambda \in [0,1],$$

i.e., $a \in \mathbb{M}(x,z)$. Therefore, $\mathbb{M}(x,z)$ is convex.

Third, we prove that $\mathbb{M}$ is upper semi-continuous with compact values. Indeed, since A is a compact set, by Lemma 1.1(ii), we need only to show that $\mathbb{M}$ is a closed mapping. In fact, assume that a net $\{(x_\alpha, z_\alpha, a_\alpha)\} \subset A \times L(X,Z) \times K(x)$ with $a_\alpha \in \mathbb{M}(x_\alpha, z_\alpha)$ such that $x_\alpha \to x \in A, z_\alpha \to z \in L(X,Z)$ and $a_\alpha \to a_0$.

Now, we need to verify that $a_0 \in \mathbb{M}(x,z)$. Since $a_\alpha \in K(x_\alpha)$ and K is upper semi-continuous on A with nonempty compact values, it follows that K is closed and so we have $a_0 \in K(x)$. Suppose that $a_0 \notin \mathbb{M}(x,z)$. There exists $y_0 \in K(x)$ such that

$$\langle z_0, \eta(y_0, a_0)\rangle \notin -\mathrm{int}C_1. \tag{2.1}$$

It follows from the lower semi-continuity of K that there is a net $\{y_\alpha\}$ such that $y_\alpha \in K(x_\alpha)$ and $y_\alpha \to y_0$ (taking a subnet if necessary). Since $a_\alpha \in \mathbb{M}(x_\alpha, z_\alpha)$, we have

$$\langle z_\alpha, \eta(y_\alpha, a_\alpha)\rangle \in Z \setminus -\mathrm{int}C_1 \text{ for all } \alpha. \tag{2.2}$$

By the condition (vi) together with (2.2), it follows that

$$\langle z, \eta(y_0, a_0)\rangle \in Z \setminus -\mathrm{int}C_1. \tag{2.3}$$

This is the contradiction from (2.1) and (2.3). Therefore, we conclude that $a_0 \in \mathbb{M}(x,z)$. Hence $\mathbb{M}$ is upper semi-continuous with nonempty compact values.

Fourth, we need to prove the solution set $\mathbb{Q}(K,T) \neq \emptyset$.

Define the set-valued mapping $\Psi : A \times L(X,Z) \rightrightarrows A \times L(X,Z)$ by

$$\Psi(x,z) = (\mathbb{M}(x,z), T(x)), \forall (x,z) \in A \times L(X,Z).$$

Then, Ψ is upper semicontinuous on $A \times L(X,Z)$, $\Psi(x,z)$ is nonempty closed convex subset of $A \times L(X,Z)$. By Lemma 1.4, there exists a point $(x,z) \in A \times L(X,Z)$ such that $(x, z) \in \Psi(x, z)$, i.e., $x \in \mathbb{M}(x, z)$, $z \in T(x^*)$. This implies that $(x, z) \in A \times T(x)$ satisfy $x \in K(x)$ and

$$\langle z, \eta(y, x)\rangle \in Z \setminus -\text{int}C_1, \forall y \in K(x),$$

i.e., the weak vector quasi-variational inequality problem has a solution.

Finally, we prove that $\mathbb{Q}(K, T)$ is compact. In fact, since A is compact and $\mathbb{Q}(K, T) \subset A$, we need only prove that $\mathbb{Q}(K, T)$ is closed. Indeed, let a net $\{x_\alpha\} \subset \mathbb{Q}(K, T)$ be such that $x_\alpha \to x_0$. Now, we prove that $x_0 \in \mathbb{Q}(K, T)$.
For any $y_0 \in K(x_0)$, it follows from the lower semi-continuity of K, there is a net $\{y_\alpha\} \subset A$ with $y_\alpha \in K(x_\alpha)$ and $y_\alpha \to y_0$. Since $x_\alpha \in \mathbb{Q}(K, T)$, there exists $z_\alpha \in T(x_\alpha)$ such that

$$\langle z_\alpha, \eta(y_\alpha, x_\alpha)\rangle \in Z \setminus -\text{int}C_1 \text{ for all } \alpha.$$

It follows from the upper semi-continuity and compactness T that $z_0 \in T(x_0)$ such that $z_\alpha \to z_0$ (taking subnets if necessary). By the condition (v) together with $(x_\alpha, y_\alpha, z_\alpha) \to (x_0, y_0, z_0)$, we have

$$\langle z_0, \eta(y_0, x_0)\rangle \in Z \setminus -\text{int}C_1,$$

this means that $x_0 \in \mathbb{Q}(K, T)$. Thus $\mathbb{Q}(K, T)$ is a closed set. Therefore, $\mathbb{Q}(K, T)$ is compact. This completes the proof. □

We now investigate the existence conditions for the weak bilevel vector variational inequality problems.

Theorem 2.1 *Suppose that all the conditions in Lemma 2.1 are satisfied,* $\mathbb{Q}(K, T))$ *is convex. Let* P *be a real locally convex Hausdorff topological vector space,* $L(X, P)$ *be the space of all linear continuous operators from* X *into* P*,* $C_2 \subset P$ *be a closed convex and pointed cone with* $\text{int}C_2 \neq \emptyset$ *and* $H : A \to L(X, P)$ *be a single-valued convex mapping. Denoted* $\langle z, x\rangle$ *by the value of a linear operator* $z \in L(X; P)$ *at* $x \in A$*, we always assume that* $\langle ., .\rangle : L(X; P) \times A \to P$ *is continuous and the following additional conditions:*

(i') *for all* $x \in \mathbb{Q}(K, T)$, $\langle H(x), x - x\rangle \in P \setminus -\text{int}C_2$*;*

(ii') *the set* $\{y \in \mathbb{Q}(K, T) : \langle H(x), y^* - x\rangle \in -\text{int}C_2\}$ *is convex;*

(iii') *for all* $y \in \mathbb{Q}(K,T)$*, the map* $x \mapsto \langle H(x), y-x\rangle$ *is weakly* C_2*-quasiconvex, i.e., for all* x_1*,* $x_2 \in \mathbb{Q}(K,T)$ *and all* $\lambda \in [0,1]$*,* $y \in \mathbb{Q}(K,T)$*, we have*

$$\langle H(x_1), y - x_1\rangle \in P \setminus -\text{int}C_2 \text{ and } \langle H(x_1), y - x_1\rangle \in P \setminus -\text{int}C_2$$
$$\Longrightarrow \langle H(\lambda x_1 + (1-\lambda)x_2), y - (\lambda x_1 + (1-\lambda)x_2)\rangle \in P \setminus -\text{int}C_2;$$

(iv') *the set* $\{(x,y) \in \mathbb{Q}(K,T) \times \mathbb{Q}(K,T) : \langle H(x), y-x\rangle \in P \setminus -\text{int}C_2\}$ *is closed.*

Then the weak bilevel vector variational inequality problem has a solution, i.e., there exists $\bar{x} \in A$ *such that* $\bar{x} \in \mathbb{Q}(K,T)$ *and*

$$\langle H(x), y - x\rangle \in P \setminus -\text{int}C_2, \;\; \forall y \in \mathbb{Q}(K,T).$$

Moreover, the solution set of the weak bilevel vector variational inequality problem is compact.

Proof. We define a multifunction $\mathbb{B} : A \rightrightarrows A$ by

$$\mathbb{B}(x) = \{b \in \mathbb{Q}(K,T) \mid \langle H(b), y - b\rangle \in P \setminus -\text{int}C_2, \;\; \forall y \in \mathbb{Q}(K,T)\}, \quad x \in A$$

First, we prove that $\mathbb{B}(x)$ is nonempty. Indeed, for all $y \in A$, $\mathbb{Q}(K,T)$ is a nonempty compact convex set. Set

$$\mathbb{P} = \{(b,y) \in \mathbb{Q}(K,T) \times \mathbb{Q}(K,T) : \langle H(b), y - b\rangle \in -\text{int}C_2\}.$$

Then we have the following: (a) The condition (i') implies that, for any $b \in \mathbb{Q}(K,T)$, $(b,b) \notin \mathbb{P}$. (b) The condition (ii') implies that, for any $b \in \mathbb{Q}(K,T)$, $\{y \in A : (b,y) \in \mathbb{P}\}$ is convex on $\mathbb{Q}(K,T)$. (c) The condition (iv') implies that, for any $b \in \mathbb{Q}(K,T)$, $\{y \in \mathbb{Q}(K,T) : (b,y) \in \mathbb{P}\}$ is open on $\mathbb{Q}(K,T)$. By Lemma 1.3, there exists $b \in \mathbb{Q}(K,T)$ such that $(b,y) \notin \mathbb{P}$ for all $y \in \mathbb{Q}(K,T)$, i.e., $\langle H(b), y-b\rangle \in P \setminus -\text{int}C_2$ for all $y \in \mathbb{Q}(K,T)\}$. Thus it follows that $\mathbb{B}(x)$ is nonempty.

Second, we show that $\mathbb{B}(x)$ is a convex set. In fact, let $b_1, b_2 \in \mathbb{B}(x)$ and $\lambda \in [0,1]$ and put $b = \lambda b_1 + (1-\lambda)b_2$. Since $b_1, b_2 \in \mathbb{Q}(K,T)$ and $\mathbb{Q}(K,T)$ is a convex set, we have $b \in \mathbb{Q}(K,T)$. Thus it follows that, for all $b_1, b_2 \in \mathbb{B}(x)$,

$$\langle H(b_1), y - b_1\rangle \in P \setminus -\text{int}C_2; \;\; \text{and} \;\; \langle H(b_2), y - b_2\rangle \in P \setminus -\text{int}C_2, \;\; \forall y \in \mathbb{B}(x).$$

By the condition (iii'), since $x \mapsto \langle H(x), y-x\rangle$ is weakly C_2-quasiconvex, we have

$$\langle H(\lambda b_1 + (1-\lambda)b_2), y - \lambda b_1 + (1-\lambda)b_2\rangle \in P \setminus -\text{int}C_2, \;\; \forall \lambda \in [0,1],$$

i.e., $b \in \mathbb{B}(x)$. Thus, $\mathbb{B}(x)$ is convex.

Third, we prove that $\mathbb{B}$ is upper semi-continuous on A with compact values. Indeed, since A is a compact set, by Lemma 1.1 (ii), we need only to show that $\mathbb{B}$ is a closed mapping. Let a net $\{x_\alpha\} \subset A$ be such that $x_\alpha \to x \in A$ and let $b_\alpha \in \mathbb{B}(x_\alpha)$ be such that $b_\alpha \to b_0$.

Now, we need to show that $b_0 \in \mathbb{B}(x)$. Since $b_\alpha \in \mathbb{Q}(K,T)$ and $\mathbb{Q}(K,T)$ is compact, we have $b_0 \in \mathbb{Q}(K,T)$. Suppose that $b_0 \notin \mathbb{B}(x)$. Then there exists $y \in \mathbb{Q}(K,T)$ such that

$$\langle H(b_0), y - b_0 \rangle \in -\mathrm{int}C_2. \tag{2.4}$$

On the other hand, since $b_\alpha \in \mathbb{B}(x_\alpha)$, we have

$$\langle H(b_\alpha), y - b_\alpha \rangle \in P \setminus -\mathrm{int}C_2 \quad \text{for all } \alpha. \tag{2.5}$$

By the condition (iv') together with (2.5), it follows that

$$\langle H(b_0), y - b_0 \rangle \in P \setminus -\mathrm{int}C_2, \tag{2.6}$$

which is a contradiction from (2.4) and (2.6). Thus $b_0 \in \mathbb{B}(x)$. Hence $\mathbb{B}$ is upper semi-continuous on A with nonempty compact values.

Fourth, we prove that the solution set $\mathbb{O}(H)$ is nonempty. In fact, since $\mathbb{B}$ is upper semi-continuous on A with nonempty compact values, by Lemma 1.4, there exists a point $\hat{x} \in A$ such that $\hat{x} \in \mathbb{B}(\hat{x})$. Hence there exists $\hat{x} \in \mathbb{Q}(K,T)$ such that

$$\langle H(\hat{x}), y - \hat{x} \rangle \in P \setminus -\mathrm{int}C_2, \ \ \forall y \in \mathbb{Q}(K,T),$$

i.e., the problem (WBVIP) has a solution.

Finally, we prove that $\mathbb{O}(H)$ is compact. Indeed, let a net $\{x_\alpha\} \subset \mathbb{O}(H)$ be such that $x_\alpha \to x_0$. Now, we prove that $x_0 \in \mathbb{O}(H)$. By the closedness of $\mathbb{Q}(K,T)$, we have $x_0 \in \mathbb{Q}(K,T)$. Since $x_\alpha \in \mathbb{O}(H)$, we obtain $x_\alpha \in \mathbb{Q}(K,T)$ and

$$\langle H(x_\alpha), y - x_\alpha \rangle \in P \setminus -\mathrm{int}C_2, \ \ \forall y \in \mathbb{Q}(K,T).$$

By the condition (iv') together with $x_\alpha \to x_0$, it follows that

$$\langle H(x_0), y - x_0 \rangle \in P \setminus -\mathrm{int}C_2, \ \ \forall y \in \mathbb{Q}(K,T),$$

which means that $x_0 \in \mathbb{O}(H)$. Thus $\mathbb{O}(H)$ is a closed set. Since $\mathbb{O}(H) \subset \mathbb{Q}(K,T)$ and $\mathbb{Q}(K,T)$ is compact. It follows that $\mathbb{O}(H)$ is compact subset of A. This completes the proof. □

3 Conclusions

In this work, we have established existence conditions to a new class of bilevel weak vector variational inequality problems. To the best of our knowledge, up to now, there have not been any works on the existence conditions of solutions for bilevel weak vector variational inequality problems by using the Kakutani-Fan-Glicksberg fixed-point theorem. Thus our results, Theorem 2.1 is new.

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