NEW CONDITIONS FOR THE EXISTENCE OF SOLUTIONS TO A FRACTIONAL DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS

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Abstract

We consider a boundary value problem involving a fractional differential equation with a g-Caputo fractional derivative. This paper establishes some new criteria for the existence of solutions to the problem, differing from those obtained by previous researchers. The method is based on the construction of a novel Green's function and the application of the Schauder fixed point theorem. Examples are provided to illustrate the fundamental distinctions between our results and earlier work.

Keywords: fixed point theorem, fractional differential equation, g-Caputo fractional derivative of order α, Green's function

1. Introduction

Lyapunov (1907) proved that if $y(t)$ is a nontrivial solution of the boundary value problem

$$\begin{cases} y''(t) + r(t)y(t) = 0, a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, r \in C([a,b],\mathbb{R}). \tag{1.2}$$

Inequality (1.2) is called the Lyapunov inequality for problem (1.1) and is used to show that problem (1.1) has no nontrivial solution.

With the development of fractional derivatives, many researchers have generalized problem (1.1) by replacing the second order derivative with fractional derivatives such as the Caputo derivative, Hilfer derivative, and other types of fractional derivatives, or by modifying the boundary conditions. These generalizations have led to various results, such as those by Ferreira (2013), Ferreira (2014), Dien (2021), and others.

One of the generalizations of problem (1.1) is:

$$\begin{cases} {}_{a+}^{C}D^{\alpha,g}\left({}_{a+}^{C}D^{\beta,g}y(t)\right)+f\big(t,y(t)\big)=0, a<t<b, 0<\alpha,\beta\le 1<\ \alpha+\beta, \\ y(a)=0, y(b)=\mathfrak{h}(u), \end{cases} \tag{1.3}$$

where $g\in\ C_+^1[a,b], f\in C([a,b]\times\mathbb{R},\mathbb{R})$, and ${}_{a+}^{C}D^{\alpha,g}(.)$ denotes the g-Caputo fractional derivative of order $\alpha\in(0,1)$. Dien (2021) showed that if there exists $M>0$ such that

$$M>\frac{\psi(M)G_{\max}}{(1-\kappa)\Gamma(\alpha+\beta)}\|g'(.)\kappa_1(.)\|_{L^1(a,b)},$$

then problem (1.3) has at least one solution.

In this paper, we also study the existence of solutions to problem (1.3), but we provide a different result from that of Dien (2021). Specifically, in Theorem 3.3, we show that problem (1.3) has at least one solution if there exists $M>0$ such that

$$M>\frac{\psi(M)}{(1-\kappa)\Gamma(\alpha+\beta)}\left\|\big(g(b)-g(.)\big)^{\alpha+\beta-1}g'(.)\kappa_1(.)\right\|_{L^1(a,b)}.$$

This result is entirely different from the one previously obtained by Dien (2021), and this difference is illustrated more clearly in Examples 3.7 and 3.8.

In Proposition 3.1, we construct a Green's function that differs from those introduced by previous authors, such as Dien (2021). As a result, some properties of the Green's function also differ (see Propositions 3.2 and 3.3), which is the main reason why our existence condition in Theorem 3.4 deviates from earlier results.

2. Preliminaries

In this section, we recall some basic definitions. For convenience in writing, we denote

$$C_+^1[a,b]=\{g\in\ C^1[a,b]: g'(t)>0,\forall\, t\in[a,b]\}$$

Definition 2.1 (Podlubny, 1999). *Let $\phi\in C^n[a,b], n\in N^+$, and α ∈ (n-1, n), then the Caputo fractional derivative of order α is the expression*

$${}_{a}^{C}D^{\alpha}\phi(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\phi^{(n)}(s)\mathrm{d}s,$$

where Γ(.) is the Gamma function.

Definition 2.2 (Osler, 1970). *Let α >0, $g\in C_+^1[a,b]$, and $\phi\in C^1[a,b]$. The fractional integral of a function ϕ with respect to the function g is defined by*

$$I_{a+}^{\alpha,g}\phi(t)=\frac{1}{\Gamma(\alpha)}\int_a^t[g(t)-g(s)]^{\alpha-1}g'(s)\phi(s)\mathrm{d}s.$$

Definition 2.3 (Almeida, 2017). *Let α >0, $n\in N^+$; $g,\phi\in C^n[a,b]$ two functions such that g'(t)>0, ∀t∈[a, b]. The g-Caputo fractional derivative of ϕ of order α is given by*

$${}_{a+}^{C}D^{\alpha,g}\phi(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t[g(t)-g(s)]^{n-\alpha-1}g'(s)\left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n\phi(s)\mathrm{d}s.$$

For $g(t) = t, \forall t \in [a,b]$, the g-Caputo fractional derivative ${}_{a+}^{C}D^{\alpha,g}(.)$ is becomes the Caputo fractional derivative ${}_{a}^{C}D^{\alpha}(.)$.

Lemma 2.4 (Almeida, 2017). *Let $n \in N^+, n-1 < \alpha < n$, and $g \in C_+^1[a,b]$, we have*

$$\left(I_{a+}^{\alpha,g}\, {}_{a+}^{C}D^{\alpha,g}\phi\right)(t) = \phi(t) + \sum_{k=0}^{n-1} c_k[g(t)-g(a)]^k,\ c_k \in R, (k = 0, \dots, n-1).$$

3. Main Results

3.1. The Green's Function and Its Properties

Proposition 3.1. *Let $0 < \alpha, \beta \le 1$ with $\alpha + \beta > 1$, and let $g \in C_+^1[a,b]$. If $y(t)$ is a solution of problem (1.3) such that ${}_{a+}^{C}D^{\beta,g}y(.), f(.,y(.)) \in C^1[a,b]$, then $y(t)$ is a solution of the following integral equation*

$$y(t) = \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} \mathfrak{h}(y) + \int_a^b G(t,s)\big(g(b)-g(s)\big)^{\alpha+\beta-1} g'(s) f(s,y(s))\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)}\left[\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} - \left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\right], a \le s < t \le b, \\ \frac{1}{\Gamma(\alpha+\beta)}\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta}, a \le t \le s \le b \end{cases}.$$

Proof. The proof is similar to that in Lemma 3.1 of Dien (2021). ∎

The function $G(t,s)$ constructed in Proposition 3.1 is called the Green's function. This construction differs from the one given by Dien (2021, Lemma 3.1), which leads to several properties of the Green's function that also differ from those in Dien (2021, Proposition 3.3 and 3.4).

Proposition 3.2. Let $0 < \alpha, \beta \le 1$,with $\alpha + \beta > 1$, $g \in C_+^1[a,b]$, and let the Green's function $G(t,s)$ be defined as in Proposition 3.1. Then

$$\max_{t,s\in[a,b]} |G(t,s)| = \frac{1}{\Gamma(\alpha+\beta)}.$$

Proof. First, we consider the case $a \le t \le s \le b$, then $G(t,s) = \frac{1}{\Gamma(\alpha+\beta)}\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta}$. Since $g \in C_+^1[a,b]$, we have

$$0 \le \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} \le 1,$$

and

$$\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} = 1 \Leftrightarrow t = s = b.$$

Therefore,

$$0 \le G(t,s) \le \frac{1}{\Gamma(\alpha+\beta)}, \text{ and } G(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \text{ if and only if } t = s = b.$$

Next, we consider $a \le s < t \le b$, and define

$$h(t,s) = \left[\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} - \left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\right].$$

By fixing t and differentiating $h(t,s)$ with respect to s, we get

$$\frac{\partial h}{\partial s}(t,s) = (\alpha+\beta-1)\frac{g'(s)[g(b)-g(t)]}{[g(b)-g(s)]^2}\left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-2} \ge 0.$$

Thus, for $a \le s < t \le b$, one has

$$1 \ge h(t,t) > h(t,s) \ge h(t,a).$$

Note that $0 < \alpha, \beta \le 1 < \alpha+\beta$, this gives

$$h(t,a) = \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta}\left[1 - \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha-1}\right] \le 0, (a < t),$$

and

$$h(t,a) = \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha+\beta-1}\left[\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{1-\alpha} - 1\right] \ge -1.$$

Therefore, $1 \ge |h(t,s)| \ge 0$, which implies that

$$\max_{a\le s<t\le b} |G(t,s)| = \frac{1}{\Gamma(\alpha+\beta)}.$$

Hence,

$$\max_{s,t\in[a,b]} |G(t,s)| = \frac{1}{\Gamma(\alpha+\beta)}. \quad ¡$$

Proposition 3.3. *Let* $0 < \alpha, \beta \le 1 < \alpha+\beta, g \in C_+^1[a,b]$, *and let the Green's function* $G(t,s)$ *be defined as in* Proposition 3.1. *Then, for* $a \le t_1 \le t_2 \le b$, *the following estimate holds:*

$$|G(t_2,s) - G(t_1,s)| \le \frac{3}{\Gamma(\alpha+\beta)}\left(\frac{g(t_2)-g(t_1)}{g(b)-g(s)}\right)^{\beta}.$$

Proof. We first note the following observations:

i) $0 < \alpha + \beta - 1 \le \beta \le 1$.

ii) $|a^p - b^p| \le |a-b|^p$, for $a, b \ge 0, p \in (0,1]$.

iii) $\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} \ge \left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}$, for $a \le s < t \le b$.

To verify iii), observe that for $a \le s < t \le b$,

$$\frac{g(t)-g(a)}{g(b)-g(a)}-\frac{g(t)-g(s)}{g(b)-g(s)}=\frac{\big(g(b)-g(t)\big)\big(g(s)-g(a)\big)}{\big(g(b)-g(a)\big)\big(g(b)-g(s)\big)}\geq 0.$$

Thus,

$$\frac{g(t)-g(a)}{g(b)-g(a)}\geq\frac{g(t)-g(s)}{g(b)-g(s)}>0,$$

which implies that

$$\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta}\geq\left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\geq\left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}.$$

We now consider three cases:

Case 1: $a\leq t_1\leq t_2\leq s\leq b$. In this case, by using ii), we get

$$|G(t_2,s)-G(t_1,s)|=\frac{1}{\Gamma(\alpha+\beta)}\left|\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}-\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}\right|\leq$$
$$\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)}\right)^{\beta}.$$

Case 2: $a\leq t_1\leq s<t_2\leq b$, by using ii) and iii), one has

$$|G(t_2,s)-G(t_1,s)|$$
$$=\frac{1}{\Gamma(\alpha+\beta)}\left|\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}-\left(\frac{g(t_2)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}-\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}\right|,$$
$$\leq\frac{1}{\Gamma(\alpha+\beta)}\left(\left|\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}-\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}\right|+\left(\frac{g(t_2)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\right),$$
$$\leq\frac{1}{\Gamma(\alpha+\beta)}\left(\left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)}\right)^{\beta}+\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}\right),$$
$$\leq\frac{2}{\Gamma(\alpha+\beta)}\left(\frac{g(t_2)-g(t_1)}{g(b)-g(s)}\right)^{\beta}.$$

Case 3: $a\leq s\leq t_1\leq t_2\leq b$. We have:

$$|G(t_2,s)-G(t_1,s)|$$
$$=\frac{1}{\Gamma(\alpha+\beta)}\left|\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}-\left(\frac{g(t_2)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}-\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}+\left(\frac{g(t_1)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\right|,$$
$$\leq\frac{1}{\Gamma(\alpha+\beta)}\left|\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}-\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}\right|$$
$$+\frac{1}{\Gamma(\alpha+\beta)}\left(\left(\frac{g(t_2)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}+\left(\frac{g(t_1)-g(s)}{g(b)-g(s)}\right)^{\alpha+\beta-1}\right),$$
$$\leq\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)}\right)^{\beta}+\frac{1}{\Gamma(\alpha+\beta)}\left(\left(\frac{g(t_2)-g(a)}{g(b)-g(a)}\right)^{\beta}+\left(\frac{g(t_1)-g(a)}{g(b)-g(a)}\right)^{\beta}\right),$$
$$\leq\frac{3}{\Gamma(\alpha+\beta)}\left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)}\right)^{\beta}.$$

Therefore, the desired estimate holds in all cases. ¡

3.2. Existence Conditions for the Integral Equation

In this section, we investigate conditions for the existence of solutions to the integral equation of the form

$$y(t) = \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} \mathfrak{h}(y) + \int_a^b G(t,s)\big(g(b)-g(s)\big)^{\alpha+\beta-1} g'(s) f(s,y(s))\mathrm{d}s, \quad (1.4)$$

in the Banach space $C[a,b]$ with the uniform norm. We impose the following assumptions:

i) $(\mathcal{A}_1)$**:** The function $\mathfrak{h}: \mathbb{R} \to \mathbb{R}$ is continuous, and there exists $\kappa \in (0,1)$ such that

$$|\mathfrak{h}(x)| \le \kappa|x|, \forall x \in \mathbb{R}.$$

ii) $(\mathcal{A}_2)$**:** The function $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist functions $\kappa_1, \kappa_2 \in C([a.b], \mathbb{R})$, and non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x)| \le \kappa_1(t).\psi(|x|), \forall t \in (a,b), \forall x \in \mathbb{R},$$

$$|f(t,x) - f(t,y)| \le \kappa_2(t).\phi(x,y), \forall t \in (a,b), \forall x,y \in \mathbb{R},$$

where $\phi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\phi(x,y) \to 0$ khi $|x-y| \to 0$.

We obtain the following existence result.

Theorem 3.4 (Existence of Solution). *Assume* $0 < \alpha, \beta \le 1$, *with* $\alpha + \beta > 1$. *Let* $g \in C_+^1[a,b]$, *and suppose that assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *are satisfied. Further assume*

$$\big(g(b) - g(.)\big)^{\alpha+\beta-1} g'(.)\kappa_i(.) \in L^1(a,b), i = 1,2.$$

If there exists $M > 0$ *such that*

$$M > \frac{\psi(M)}{(1-\kappa)\Gamma(\alpha+\beta)} \left\| \big(g(b) - g(.)\big)^{\alpha+\beta-1} g'(.)\kappa_1(.) \right\|_{L^1(a,b)}, \quad (1.5)$$

then the integral equation (1.4) has at least one solution $y_0 \in C[a,b]$.

Proof. Consider the operator $S: C[a,b] \to C[a,b]$ defined by

$$(Sy)(t) = \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\beta} \mathfrak{h}(y) + \int_a^b G(t,s)\big(g(b)-g(s)\big)^{\alpha+\beta-1} g'(s) f\big(s,y(s)\big)\mathrm{d}s,$$

where $G(t,s)$ is the Green's function determined by the boundary conditions of the problem (1.3).

We show that S satisfies the conditions of the Schauder fixed point theorem.

- **Continuity of S**

Let $y, z \in B_r = \{u \in C[a,b]: \|u\| \le r\}, (r > 0)$. Then:

$$\|Sy - Sz\| \le \kappa\|y - z\| + \frac{1}{\Gamma(\alpha+\beta)} \|\phi(y(.), z(.))\|. \left\| \big(g(b) - g(.)\big)^{\alpha+\beta-1} g'(.)\kappa_2(.) \right\|_{L^1(a,b)}.$$

Since $\phi\big(y(s), z(s)\big) \to 0$ as $y \to z$, it follows that is S continuous.

- ***S* maps bounded sets into bounded sets**

For $y \in B_r$, we have

$$\|Sy\| \le \kappa r + \frac{\psi(r)}{\Gamma(\alpha+\beta)} \left\| \big(g(b) - g(.)\big)^{\alpha+\beta-1} g'(.)\kappa_1(.) \right\|_{L^1(a,b)}.$$

- ***S* maps bounded sets into equicontinuous sets:**

For $y \in B_r, a \leq t_1 < t_2 \leq b$, we have

$$|S(y(t_2)) - S(y(t_1))| \leq \left| \left(\frac{g(t_2) - g(a)}{g(b) - g(a)} \right)^\beta - \left(\frac{g(t_1) - g(a)}{g(b) - g(a)} \right)^\beta \right| |\mathfrak{h}(y)|$$

$$+ \int_a^b |G(t_2, s) - G(t_1, s)|. (g(b) - g(s))^{\alpha+\beta-1} g'(s). |f(s, y(s))| ds,$$

$$\leq \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta || \mathfrak{h} || + \frac{3}{\Gamma(\alpha+\beta)} \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta \int_a^b (g(b) - g(s))^{\alpha+\beta-1} g'(s) |f(s, y(s))| ds,$$

$$\leq \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta \kappa r + \frac{3\psi(r)}{\Gamma(\alpha+\beta)} \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta \int_a^b (g(b) - g(s))^{\alpha+\beta-1} g'(s) \kappa_1(s) ds,$$

$$= \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta \kappa r + \frac{3\psi(r)}{\Gamma(\alpha+\beta)} \left(\frac{g(t_2)-g(t_1)}{g(b)-g(a)} \right)^\beta \left\| (g(b) - g(.))^{\alpha+\beta-1} g'(.) \kappa_1(.) \right\|_{L^1(a,b)},$$

$$= C_1 (g(t_2) - g(t_1))^\beta,$$

where C_1 is constant depending on $r, \psi(r), \kappa$, but independent of y. Since g is continuous, it follows that $S(B_r)$ is relatively compact.

- **Application of the Schauder fixed point theorem**

Suppose $y = \lambda S y$ for some $y \in \partial B_M, \lambda \in (0,1)$. Then

$$M = \|y\| \leq \|Sy\| \leq \kappa M + \frac{\psi(M)}{\Gamma(\alpha + \beta)} \left\| (g(b) - g(.))^{\alpha+\beta-1} g'(.) \kappa_1(.) \right\|_{L^1(a,b)}.$$

Thus,

$$M \leq \frac{\psi(M)}{(1 - \kappa)\Gamma(\alpha + \beta)} \left\| (g(b) - g(.))^{\alpha+\beta-1} g'(.) \kappa_1(.) \right\|_{L^1(a,b)}.$$

This contradicts the assumption (1.5). Hence, S has at least one fixed point $y_0 \in B_M$.This fixed point is a solution to the integral equation (1.4). The theorem is proved. ¡

Theorem 3.5 *([5], Theorem 4.7). Let* $0 < \alpha, \beta \leq 1 < \alpha + \beta$, *and suppose* $g \in C_+^1[a, b], \mathfrak{h} \in C(\mathbb{R}, \mathbb{R})$. *Assume that:*

i) *Conditions* $(\mathcal{A}_1)$, *and* $(\mathcal{A}_2)$ *are satisfied;*

ii) $g'(.)\kappa_i(.) \in L^1(a, b), i = 1,2$.

If there exists $M > 0$ *such that*

$$M > \frac{\psi(M) G_{max}}{(1 - \kappa)\Gamma(\alpha + \beta)} \|g'(.)\kappa_1(.)\|_{L^1(a,b)}, \tag{1.6}$$

where

$$G_{max} = (g(b) - g(a))^{\alpha+\beta-1} max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha+\beta-1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha+2\beta-1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\},$$

then the boundary value problem (1.4) has at least one solution.

Remark 3.6. Assuming the hypotheses in both Theorem 3.4 and Theorem 3.5 hold, we compare the thresholds for $M > 0$ given in inequalities (1.5) and (1.6) to show that the results of these two theorems are essentially different. Specifically, observe the distinction between the following two quantities:

- $A = \left\| \left(g(b) - g(.)\right)^{\alpha+\beta-1} g'(.)\kappa_1(.) \right\|_{L^1(a,b)}$,
- $B = \left(g(b) - g(a)\right)^{\alpha+\beta-1}. G_{\max}. \|g'(.)\kappa_1(.)\|_{L^1(a,b)}$.

We will now illustrate the difference between A and B through examples.

Example 3.7. Let $\alpha = 0,5; \beta = 1; a = 0; b = 1$, and define the functions:

- $g(t) = t, \forall t \in [0,1]$,
- $\kappa_1(t) = t, \forall t \in [0,1]$,
- $f(t,x) = t.x, \forall t \in [0,1], \forall x \in \mathbb{R}$.
- Then we compute:

$$A - B = \int_0^1 (b-s)^{0,5}\, s ds - \frac{2\sqrt{3}}{9}\int_0^1 s\, ds = \frac{4}{15} - \frac{\sqrt{3}}{9} > 0 \Rightarrow A > B.$$

Example 3.8. Again, let $\alpha = 0,5; \beta = 1; a = 0; b = 1$, and define:

- $g(t) = t, \forall t \in [0,1]$,
- $\kappa_1(t) = t^5, \forall t \in [0,1]$,
- $f(t,x) = t.x, \forall t \in [0,1], \forall x \in \mathbb{R}$.

Then

$$A - B = \int_0^1 (b-s)^{0,5}\, s^5 ds - \frac{2\sqrt{3}}{9}\int_0^1 s^5\, ds = \frac{512}{9009} - \frac{\sqrt{3}}{27} < 0 \Rightarrow A < B.$$

4. Conclusion and Further Directions

This paper presents an alternative approach to constructing the Green's function, thereby extending the existence conditions for fractional differential equations with boundary conditions. This expansion enables the application of the results to a broader class of nonlinear problems.

In terms of applications, fractional differential equations with complex boundary conditions arise in various real-world models, such as those in materials physics, systems biology, control engineering, and finance. Establishing the existence of solutions is a foundational step in ensuring the mathematical validity of such models before any quantitative analysis or numerical simulation is performed. Hence, the results in this paper may serve as a verification tool in engineering practice, particularly for systems involving non-standard or nonlinear-dependent boundary conditions.

Future work may focus on studying the uniqueness and stability of solutions, as well as generalizing the results to other types of fractional derivatives or different initial conditions. Another practical direction involves integrating the theoretical results with numerical methods to develop solution algorithms.

References

Almeida, R. (2017). A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul*, 44, 460-481.

Dien, N. M. (2021). Nonlinear sequential fractional boundary value problems involving generalized ψ-Caputo fractional derivatives. *Filomat, 15*(36)*,* 5047-5058.

Ferreira, A. M. (2013). A Lyapunov-type inequality for a fractional boundary value problem. *Fract. Calc. Appl. Anal, 16*(4), 978-984.

Ferreira, A. M. (2014). On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. *J. Math. Anal. Appl,* 412, 1058-1063.

Lyapunov, A. M. (1907). Probléme général de la stabilité du mouvement. *Ann. Fac. Sci. Univ. Toulouse,* 2, 203-407.

Osler, T. J. (1970). Fractional derivatives of a composite function. *SIAM J. Math. Anal,* 1, 288-293.

Podlubny, I. (1999). *Fractional differential equations*, Academic Press.