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Inverse source problem for Sobolev equation with observed data in L^b spaces

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ABSTRACT

In this paper, we are interested in the problem of determining the source function forthe Sobolev equation with fractional Laplacian. This problem is non-well-posed. We show theerror estimate between the exact solution and the regularized solution with the observed data in L^b spaces.

Keywords: *Sobolev equations, inverse source problem, Sobolev embeddings*

1 Introduction

For $\alpha > 1$, let $\Omega \in \mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we investigate the following problem

$$\nu_t - m\Delta\nu_t + (-\Delta)^\alpha \nu = F(x,t), \quad \text{in} \quad \Omega \times (0,T] \tag{1.1}$$

where $m > 0$ is the diffusion coefficient, F is the source function and u desribe the distribution of the temperature at space x and time t. Sobolev equation describing various physical phenomena, such as heat conduction [1], homogeneous liquid permeability [2], propagation of long waves in a nonlinearly dispersed medium [3]. From the fractional Laplacian $(-\Delta)^\alpha$ with properties of fractional operator $(-\Delta)^\alpha$ has been described in detail in [1], [4], [5], [6]. The main purpose of this paper is to determine the source function $F(x,t) = \varphi(t)f(x)$ with the split form when we know that

$$\nu(x,T) = \xi(x), \nu(x,0) = 0, \quad x \in \Omega. \tag{1.2}$$

The question of determining the function f when we know φ and ξ will be studied carefully in this paper. It is surprising that the problem of determining the source function for the pseudo-parabolic equation has not been investigated before. Our main task here is to construct a regularized method which looking for the function f_ϵ and claim that

$$\lim \|f_\epsilon - f\| = 0, \text{ when } \epsilon \to 0^+. \tag{1.3}$$

within the appropriate range. It can be affirmed that our paper is one of the first works on the inverse source problem for the Sobolev equation, on the regularization methods studied. In [7], the regular method Tikhonov. Ma-Prakash-Deiveegan [8] applied the generalization and modification Tikhonov regularization methods. The Landweber normalization method was first derived from [9], [10], [11], [12]. Binh and co-authors, see [13] studied the Rayleigh-Stokes problem through the Tikhonov method. However, these error estimates in the L^2 spaces, the results in this paper are evaluated in the L^b spaces, which also makes a new point in this paper. In addition, the source function of the current paper depends on the φ function, making the calculation more cumbersome.

This article is organized as follows. Section 2 presents some preparatory knowledge. In the section 3, we give the Fourier formula of the source function of the problem. In Section 4, we provide a convergence estimate.

2 Preliminary Results

The operator $\mathcal{A} = -\Delta$ on $\mathcal{V} = \mathcal{H}_0^1(\Omega) \cap H^2(\Omega)$, and $\mathcal{A}$ has the eigenvalues λ_n such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are denoted by $e_n \in \mathcal{V}$. Next, we define by $\mathcal{A}^\sigma$ the following operator

$$\mathcal{A}^\sigma v := \sum_{n=1}^{\infty} \lambda_n^\sigma \langle v, e_n \rangle e_n(x), \; v \in D(\mathcal{A}^\sigma) = \Big\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} |\langle v, e_n \rangle|^2 \lambda_n^{2\sigma} < \infty \Big\}. \tag{2.1}$$

The domain $D(\mathcal{A}^\sigma)$ is the Banach space equipped with the norm

$$\|v\|_{D(\mathcal{A}^\sigma)} := \Big(\sum_{n=1}^{\infty} |\langle v, e_n \rangle|^2 \lambda_n^{2\sigma} \Big)^{\frac{1}{2}} \tag{2.2}$$

Lemma 2.1. *Let assume that $\alpha > 1$, and $\varphi : [0, T] \to \mathbb{R}$ such that $0 < \mathcal{A}_1 \leq \varphi(t) \leq \mathcal{A}_2$. Then we receive*

$$\int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{1 + m\lambda_n} \right) \varphi(s) ds \leq \mathcal{A}_2 \frac{1 + m\lambda_n}{\lambda_n^\alpha}, \tag{2.3}$$

and

$$\begin{aligned} &\frac{1 + a\lambda_n}{\lambda_n^\alpha} \left(1 - \exp\left(-T \frac{\lambda_1^\alpha}{1 + m\lambda_1} \right) \right) \mathcal{A}_1 \\ &\qquad \leq \int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{(1 + m\lambda_n)} \right) \varphi(s) ds. \end{aligned} \tag{2.4}$$

Proof. See in [14]. □

Lemma 2.2. *(See [?] The following statement are true:*

$$\left. \begin{array}{llll} L^p(\Omega) \hookrightarrow D(\mathcal{A}^\mu)(\Omega), & \textit{if} & -\frac{N}{4} < \mu \leq 0, & p \geq \frac{2N}{N-4\mu} \\ D(\mathcal{A}^\mu)(\Omega) \hookrightarrow L^p(\Omega), & \textit{if} & 0 \leq \mu < \frac{N}{4}, & p \leq \frac{2N}{N-4\mu} \end{array} \right\}$$

3 Inverse source problem

Taking the inner product of both sides of (1.1) with $e_n(x)$, we get

$$\begin{aligned} &\frac{d}{dt} \Big(\int_\Omega \nu(x,t) e_n(x) dx \Big) + m\lambda_n \Big(\int_\Omega \nu(x,t) e_n(x) dx \Big) \\ &\qquad + \lambda_n^\alpha \Big(\int_\Omega \nu(x,t) e_n(x) dx \Big) = \int_\Omega F(x,t) e_n(x) dx, \end{aligned} \tag{3.1}$$

and $u(x,0)=0$, we have that

$$\int_\Omega \xi(x)e_n(x)dx = \int_0^T \exp\Big(\frac{-(T-s)\lambda_n^\alpha}{1+m\lambda_n}\Big)\Big(\int_\Omega F(x,s)e_n(x)dx\Big)ds, \tag{3.2}$$

since the fact that $F(x,t)=\varphi(t)f(x)$, we get

$$\int_\Omega f(x)e_n(x)dx = \frac{\int_\Omega \xi(x)e_n(x)dx}{\int_0^T \exp\Big(\frac{-(T-s)\lambda_n^\alpha}{1+m\lambda_n}\Big)\varphi(s)ds}. \tag{3.3}$$

Hence, we have

$$f(x) = \sum_{n=1}^{\infty}\Big(\frac{\int_\Omega \xi(x)e_n(x)dx}{\int_0^T \exp\Big(\frac{-(T-s)\lambda_n^\alpha}{1+m\lambda_n}\Big)\varphi(s)ds}\Big)e_n(x). \tag{3.4}$$

4 Regularization on L^b spaces

The following theorem will be our main result in this paper

Theorem 4.1. *Let us take $(\xi_\epsilon,\varphi_\epsilon)$ is observed data of (ξ,φ) such that*

$$\|\varphi_\epsilon-\varphi\|_{L^b(0,T)} + \|\xi_\epsilon-\xi\|_{L^b(\Omega)} \le \epsilon, \tag{4.1}$$

and $\varphi_\epsilon(t) > \mathcal{A}_3 > 0$ for any $\alpha > 1$, and $\alpha^{-1} < b < 2$. Assume that $f \in D\left(\mathcal{A}^{\sigma+k}\right)$ for $\sigma > 0$ and $0 < k < \frac{N}{4}$. A regularized solution built as follows

$$f_\epsilon^{\mathcal{C}}(x) = \sum_{n=1}^{\mathcal{C}_\epsilon} \frac{\left(\int_\Omega \xi_\epsilon(x)e_n(x)dx\right)e_n(x)}{\int_0^T \exp\Big(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\Big)\varphi_\epsilon(s)ds}. \tag{4.2}$$

Then the error estimate

$$\left\|f_\epsilon^{\mathcal{C}_\epsilon}-f\right\|_{L^{\frac{2N}{N-4k}}(\Omega)} \text{ is of order} \max\Big\{\epsilon^\delta, \epsilon^{\frac{\sigma(1-\delta)}{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}}, \epsilon^{\frac{(\alpha-1)\delta+k+\frac{N}{b}-\frac{N}{2}}{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}}\Big\}. \tag{4.3}$$

If we choice $\mathcal{C}_\epsilon$ satisfies

$$\lim_{\epsilon\to0}|\mathcal{C}_\epsilon|^{\alpha-1}\epsilon = \lim_{\epsilon\to0}\Big(|\mathcal{C}_\epsilon|^{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}\epsilon\Big) = 0, \quad \lim_{\epsilon\to0}\mathcal{C}_\epsilon = +\infty\cdot \tag{4.4}$$

Remark 4.2. *$\mathcal{C}_\epsilon$ is chosen as follows:*

$$\mathcal{C}_\epsilon = \epsilon^{\frac{\delta-1}{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}}, 0<\delta<1. \tag{4.5}$$

Proof. In view of triangle inequality, we find that

$$\left\|f_\epsilon^{\mathcal{C}_\epsilon}-f\right\|_{D(\mathcal{A}^k)} \le \left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-f\right\|_{D(\mathcal{A}^k)} + \left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-\mathcal{F}_1^{\mathcal{C}_\epsilon}\right\|_{D(\mathcal{A}^k)} + \left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-f_\epsilon\right\|_{D(\mathcal{A}^k)}, \tag{4.6}$$

whereby

$$\mathcal{F}_1^{\mathcal{C}_\epsilon}(x) = \sum_{n=1}^{\mathcal{C}_\epsilon} \frac{\left(\int_\Omega \xi(x)e_n(x)dx\right)e_n(x)}{\int_0^T \exp\Big(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\Big)\varphi_\epsilon(s)ds},$$

$$\mathcal{F}_2^{\mathcal{C}_\epsilon}(x) = \sum_{n=1}^{\mathcal{C}_\epsilon} \frac{\left(\int_\Omega \xi(x)e_n(x)dx\right)e_n(x)}{\int_0^T \exp\Big(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\Big)\varphi(s)ds}. \tag{4.7}$$

Next, we considered the upper bound of (4.6). For convenience, we consider the following step.
Step 1 Estimate of $\left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-f\right\|_{D(\mathcal{A}^k)}$

$$\begin{aligned}\mathcal{F}_2^{\mathcal{C}_\epsilon}-f &= \sum_{n=\mathcal{C}_\epsilon+1}^{\infty}\left[\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi(s)ds\right]^{-1}\left(\int_\Omega \xi(x)e_n(x)dx\right)e_n(x)\\ &= \sum_{n=\mathcal{C}_\epsilon+1}^{\infty}\left(\int_\Omega f(x)e_n(x)dx\right)e_n(x). \end{aligned} \tag{4.8}$$

From (4.8), using the Parseval equality, we have

$$\begin{aligned}\left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-f\right\|^2_{D(\mathcal{A}^k)} &= \sum_{n=\mathcal{A}_\epsilon+1}^{\infty}\lambda_n^{2k}\left(\int_\Omega f(x)e_n(x)dx\right)^2\\ &= \sum_{n=\mathcal{A}_\epsilon+1}^{\infty}\lambda_n^{-2\sigma}\lambda_n^{2\sigma+2k}\left(\int_\Omega f(x)e_ndx\right)^2. \end{aligned} \tag{4.9}$$

If $\lambda_n > \mathcal{C}_\epsilon$ and $n>0$, , it is easy to see that $\lambda_n^{-2\sigma}\le|\mathcal{C}_\epsilon|^{-2\sigma}$, this implies that

$$\begin{aligned}\left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-f\right\|^2_{D(\mathcal{A}^k)} &\le |\mathcal{C}_\epsilon|^{-2\sigma}\sum_{n=\mathcal{C}_\epsilon}^{\infty}\lambda_n^{2\sigma+2k}\left(\int_\Omega f(x)e_n(x)dx\right)^2\\ &= |\mathcal{C}_\epsilon|^{-2\sigma}\left\|f\right\|^2_{D(\mathcal{A}^{\sigma+k})}. \end{aligned} \tag{4.10}$$

It gives

$$\left\|\mathcal{F}_2^{\mathcal{C}_\epsilon}-f\right\|_{D(\mathcal{A}^{\sigma+k})} \le |\mathcal{C}_\epsilon|^{-\sigma}\left\|f\right\|_{D(\mathcal{A}^{\sigma+k})} \tag{4.11}$$

Step 2. Estimate of $\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-\mathcal{F}_2^{\mathcal{C}_\epsilon}\right\|_{D(\mathcal{A}^k)}$

$$\begin{aligned}\mathcal{F}_1^{\mathcal{C}_\epsilon}(x)-\mathcal{F}_2^{\mathcal{C}_\epsilon}(x) &= \sum_{n=1}^{\mathcal{C}_\epsilon}\frac{\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\left(\varphi_\epsilon(s)-\varphi(s)\right)ds}{\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi_\epsilon(s)ds}\\ &\qquad\times\frac{\int_\Omega \xi(x)e_n(x)dx}{\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi(s)ds}e_n(x). \end{aligned} \tag{4.12}$$

We follows from (4.12) that

$$\begin{aligned}&\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-\mathcal{F}_2^{\mathcal{C}_\epsilon}\right\|^2_{D(\mathcal{A}^k)}\\ &= \sum_{n=1}^{\mathcal{C}_\epsilon}\left[\frac{\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\left(\varphi_\epsilon(s)-\varphi(s)\right)ds}{\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi_\epsilon(s)ds}\right]^2\lambda_n^{2k}\left(\int_\Omega f(x)e_n(x)dx\right)^2. \end{aligned} \tag{4.13}$$

Thank to Hölder inequality, we know that

$$\begin{aligned}&\left|\int_0^T \exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\left(\varphi_\epsilon(s)-\varphi(s)\right)ds\right|\\ &\le\left(\int_0^1 \left|\varphi_\epsilon(s)-\varphi(s)\right|^r ds\right)^{\frac{1}{b}}\left(\int_0^T \exp\left(-b^*(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)ds\right)^{\frac{1}{b^*}}, \end{aligned} \tag{4.14}$$

where $b^* = \frac{b}{b-1}$. It provide the following statement

$$\left(\int_0^T |\varphi_\epsilon(s) - \varphi(s)|^b \, ds \right)^{\frac{1}{b}} = \|\varphi_\epsilon - \varphi\|_{L^b(0,T)} . \tag{4.15}$$

Next, using the inequality $1 - \exp(-x) \le x, \forall x \ge 0$, one has

$$\begin{aligned} \left(\int_0^T \exp\left(-b^*(T-s) \frac{\lambda_n^\alpha}{1+m\lambda_n} \right) ds \right) &= \frac{1}{r^* \frac{\lambda_n^\alpha}{1+m\lambda_n}} \left(1 - \exp\left(-Tb^* \frac{\lambda_n^\alpha}{1+m\lambda_n} \right) \right) \\ &\le T. \end{aligned} \tag{4.16}$$

where we note that $b > \alpha^{-1}$, combining three evaluations (4.14), (4.15), and (4.16), we derive that the following estimate

$$\left| \int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{1+m\lambda_n} \right) (\varphi_\epsilon(s) - \varphi(s)) \, ds \right| \le T^{\frac{1}{b^*}} \|\varphi_\epsilon - \varphi\|_{L^b(0,T)} . \tag{4.17}$$

Next, let assume that φ_ϵ by a positive constant $\mathcal{A}_3$, we have immediately

$$\begin{aligned} &\int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{(1+m\lambda_n)} \right) \varphi_\epsilon(s) ds \\ &\ge \frac{1+a\lambda_n}{\lambda_n^\alpha} \left(1 - \exp\left(-T \frac{\lambda_1^\alpha}{1+m\lambda_1} \right) \right) \mathcal{A}_3 \\ &\ge \frac{1}{\lambda_n^{\alpha-1}} a\mathcal{A}_3 \left(1 - \exp\left(-T \frac{\lambda_1^\alpha}{1+m\lambda_1} \right) \right) . \end{aligned} \tag{4.18}$$

We assert that

$$\begin{aligned} &\frac{\int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{1+m\lambda_n} \right) (\varphi_\epsilon(s) - \varphi(s)) \, ds}{\int_0^T \exp\left(-(T-s) \frac{\lambda_n^\alpha}{1+m\lambda_n} \right) \varphi_\epsilon(s) ds} \\ &\le \lambda_n^{\alpha-1} T^{\frac{1}{b^*}} \|\varphi_\epsilon - \varphi\|_{L^b(0,T)} \left[a\mathcal{A}_3 \left(1 - \exp\left(-T \frac{\lambda_1^\alpha}{1+m\lambda_1} \right) \right) \right]^{-1} . \end{aligned} \tag{4.19}$$

By we denote

$$\mathcal{A}_5 = T^{\frac{1}{b^*}} \left[a\mathcal{A}_3 \left(1 - \exp\left(-T \frac{\lambda_1^\alpha}{1+m\lambda_1} \right) \right) \right]^{-1} . \tag{4.20}$$

Combining (4.14) to (4.20), we can see that

$$\left\| \mathcal{F}_1^{\mathcal{C}_\epsilon} - \mathcal{F}_2^{\mathcal{C}_\epsilon} \right\|_{D(\mathcal{A}^k)}^2 \le \mathcal{A}_5^2 \sum_{n=1}^{\mathcal{C}_\epsilon} \lambda_n^{2\alpha-2+2k} \left(\int_\Omega f(x) e_n(x) dx \right)^2 . \tag{4.21}$$

The finite sum $\sum_{n=1}^{\mathcal{C}_\epsilon} \lambda_n^{2\alpha-2+2k} \left(\int_\Omega f(x) e_n(x) dx \right)^2$ is bounded by

$$|\mathcal{C}_\epsilon|^{2\alpha-2} \sum_{n=1}^{\mathcal{C}_\epsilon} \lambda_n^{2k} \left(\int_\Omega f(x) e_n(x) dx \right)^2 \le |\mathcal{C}_\epsilon|^{2\alpha-2} \|f\|_{D(\mathcal{A}^k)}^2 . \tag{4.22}$$

Therefore, we follows from (4.22) that

$$\left\| \mathcal{F}_1^{\mathcal{C}_\epsilon} - \mathcal{F}_2^{\mathcal{C}_\epsilon} \right\|_{D(\mathcal{A}^k)} \le \mathcal{A}_5 \|f\|_{D(\mathcal{A}^k)} |\mathcal{C}_\epsilon|^{\alpha-1} \epsilon. \tag{4.23}$$

Step 3. Estimate of $\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-f_\epsilon^{\mathcal{C}_\epsilon}\right\|_{D\left(\mathcal{A}^k\right)}$. From (4.12) and (4.16), we receive

$$\mathcal{F}_1^{\mathcal{C}_\epsilon}(x)-f_\epsilon^{\mathcal{C}_\epsilon}(x)=\sum_{n=1}^{\mathcal{C}_\epsilon}\frac{\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)dx}{\int_0^T\exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi_\epsilon(s)ds}e_n(x). \tag{4.24}$$

From (4.24), by taking the norm in space $D\left(\mathcal{A}^k\right)$, add Parseval' s equality, we obtain that

$$\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-f_\epsilon^{\mathcal{C}_\epsilon}\right\|^2_{D\left(\mathcal{A}^k\right)}=\sum_{n=1}^{\mathcal{C}_\epsilon}\left(\frac{\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)dx}{\int_0^T\exp\left(-(T-s)\frac{\lambda_n^\alpha}{1+m\lambda_n}\right)\varphi_\epsilon(s)ds}\right)^2. \tag{4.25}$$

Thank to the inequality (4.18), we get

$$\begin{aligned}\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-f_\epsilon^{\mathcal{C}_\epsilon}\right\|^2_{D\left(\mathcal{A}^k\right)}=&\left[a\mathcal{A}_3\left(1-\exp\left(\frac{-T\lambda_1^\alpha}{1+m\lambda_1}\right)\right)\right]^{-2}\\&\times\sum_{n=1}^{\mathcal{C}_\epsilon}\lambda_n^{2\alpha-2+2k}\left(\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)dx\right)^2.\end{aligned} \tag{4.26}$$

Since $1<b<2$, we know that $L^b(\Omega)\hookrightarrow D(\mathcal{A}^{\frac{Nr-2N}{2r}})(\Omega)$. we continue to deal with the finite series on the right above as follows

$$\begin{aligned}&\sum_{n=1}^{\mathcal{C}_\epsilon}\lambda_n^{2\alpha-2+2k}\left(\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)(x)dx\right)^2\\&\quad=\sum_{n=1}^{\mathcal{C}_\epsilon}\lambda_n^{2\alpha-2+2k+\frac{2N}{b}-N}\lambda_n^{\frac{Nb-2N}{b}}\left(\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)dx\right)^2\\&\le\left|\mathcal{C}_\epsilon\right|^{2\alpha-2+2k+\frac{2N}{b}-N}\sum_{n=1}^{\mathcal{C}_\epsilon}\lambda_n^{\frac{Nb-2N}{b}}\left(\int_\Omega\left(\xi_\epsilon(x)-\xi(x)\right)e_n(x)dx\right)^2\\&=\left|\mathcal{C}_\epsilon\right|^{2\alpha-2+2k+\frac{2N}{b}-N}\left\|\xi_\epsilon-\xi\right\|_{D(\mathcal{A}^{\frac{Nb-2N}{2b}})}\lesssim\left|\mathcal{C}_\epsilon\right|^{2\alpha-2+2k+\frac{2N}{b}-N}\left\|\xi_\epsilon-\xi\right\|_{L^b(\Omega)}.\end{aligned} \tag{4.27}$$

By summarizing all three evaluations 4.32, 4.33, we infer that

$$\left\|\mathcal{F}_1^{\mathcal{C}_\epsilon}-f_\epsilon^{\mathcal{C}_\epsilon}\right\|_{D\left(\mathcal{A}^k\right)}\le\mathcal{A}_5T^{-\frac{1}{b^*}}\left|\mathcal{C}_\epsilon\right|^{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}\epsilon. \tag{4.28}$$

From three steps, we can conclude that

$$\begin{aligned}\left\|f_\epsilon^{\mathcal{C}_\epsilon}-f\right\|_{D\left(\mathcal{A}^k\right)}\le&\left|\mathcal{C}_\epsilon\right|^{-\sigma}\|f\|_{D\left(\mathcal{A}^{\sigma+k}\right)}+\mathcal{A}_5\|f\|_{D\left(\mathcal{A}^k\right)}\left|\mathcal{C}_\epsilon\right|^{\alpha-1}\epsilon\\&+\mathcal{A}_5T^{-\frac{1}{b^*}}\left|\mathcal{C}_\epsilon\right|^{\alpha-1+k+\frac{N}{b}-\frac{N}{2}}\epsilon.\end{aligned} \tag{4.29}$$

By using Lemma 2.2, since $0<k<\frac{N}{4}$, we know that $D\left(\mathcal{A}^k\right)\hookrightarrow L^{\frac{2N}{N-4k}}(\Omega)$, which yields to the desired result (4.3). □

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