EXISTENCE AND UNIQUENESS RESULTS FOR PANTOGRAPH DIFFERENTIAL EQUATIONS CONNECTED TO RIESZ-CAPUTO FRACTIONAL DERIVATIVE WITH A WEAKLY SINGULAR SOURCE

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Abstract

Fractional differential equations are an important branch of mathematics and have been considered under many different fractional derivatives. Among them, differential equations with Riesz-Caputo fractional derivatives have also attracted the attention of many researchers. Studying differential equations that may have singularity coefficients is more difficult than usual because they require several complex techniques. In the present paper, we consider a nonlinear pantograph differential equation where the source function may have a temporal singularity. Using the contraction principle, we prove that the problem has a unique solution under some appropriate conditions. Furthermore, we define a new type of Ulam-Hyers stability and show the main equation of the problem is stable in the mentioned sense. To obtain the main results, a new inequality is proposed and proved. Some examples are constructed to confirm the validity and feasibility of the theoretical results.

Keywords: contraction principle, Fractional differential equations, fractional pantograph equations, Riesz-Caputo fractional derivatives, Ulam-Hyers stability

1. Introduction

1.1. Background and literature

Fractional calculus is a significant branch of mathematics that provides tools to model physical problems involving memory effects (Goncalves et al, 2020). Nowadays, applications of fractional calculus have been found in many fields of science and engineering, such as chemistry, electrical engineering, quantum mechanics, the fluid-dynamic traffic model, electro-dynamics, pollution control, turbulence, etc., we refer to (Coffey et al., 2004; Iomin, 2019; Magin, 2006; Tarasov, 2010). Fractional derivatives are one of the central concepts of fractional calculus. Numerous concepts of fractional derivatives have been studied in literature such as Caputo, Hadamard, Riemann-Liouville, Hilfer, and Riesz-Caputo.

Various models have been proposed and studied for each type of derivative. Problems involving Riesz-Caputo fractional derivatives have also been considered in many papers. For instance, the works (Adjimi et al., 2022; Agrawal, 2007; Aleem et al., 2021; Chen et al., 2017; Chen et al., 2019; Rahou et al., 2023; Toprakseven, 2023) have considered the existence and uniqueness results for fractional differential equations with various initial/boundary value conditions. Gu et al. (Gu et al., 2019) established some criteria for the fractional differential equations with the Riesz space derivative to have a positive solution. It is worth noting that in the mentioned works, source functions of problems were assumed continuous. Problems related to Riesz-Caputo fractional derivatives with singularities have not been studied yet. Furthermore, pantograph problems with the mentioned derivative have also not been examined.

1.2. The aim of the paper

Motivated by the above analysis, we consider the pantograph problem involving Riesz-Caputo fractional derivative as follows

$$\begin{cases} {}^{RC}_{0}D^{\alpha}_{T}u(t) = f\big(t, u(t), u(\lambda t)\big), \qquad 0 < \alpha \le 1, \lambda \in (0,1), t \in (0,T), \\ u(0) = A, u(T) = B. \end{cases} \tag{1}$$

Unlike the previous work, herein, we consider the source function of the problem may have a singularity. We provide some conditions such that the problem processes a solution uniquely. Moreover, we demonstrate that the main equation is Ulam-Hyers stability in a new sense.

1.3. Outline of the paper

The remainder of the paper is structured as follows. Section 2 presents some definitions and lemmas. Section 3 is devoted to stating and proving the main results. Section 4 provides some examples to show the applicability of theoretical results. Some conclusions are given in Section 5.

2. Mathematical preliminaries

This part introduces some denotes, definitions, and lemmas that are the background knowledge for this study.

Throughout this paper, we denote $C[0,T]$ the space of all continuous functions on $[0,T]$. For $u \in C[0,T]$, let us denote $\|u\| = \sup\limits_{0\le t\le T} |u(t)|$.

We continue by giving some definitions of fractional integral and Riesz-Caputo fractional derivatives

Definition 1. *Let $n \in N$ and $\alpha \in (n-1, n]$. The Riesz-Caputo fractional derivative is defined (Kilbas et al., 2006) as follows*

$${}^{RC}_{0}D^{\alpha}_{T}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^T |t-\tau|^{\alpha-n}u^{(n)}(\tau)d\,\tau$$

$$= \frac{1}{2}\left({}^{C}_{0}D^{\alpha}_{t} + (-1)^n\, {}^{C}_{t}D^{\alpha}_{T}\right)u(t),$$

where $\Gamma(.)$ is the Gamma function,

$$ {}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{\alpha-n} u^{(n)}(\tau) d\,\tau $$

and

$$ {}_t^C D_T^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T (\tau-t)^{\alpha-n} u^{(n)}(\tau) d\,\tau. $$

In particularly, if $\alpha \in (0,1]$ *and* $u \in C(0,T)$, *then*

$$ {}^{RC}_0 D_T^\alpha u(t) = \frac{1}{2} ({}^{RC}_0 D_t^\alpha - {}^{RC}_t D_T^\alpha) u(t). $$

Definition 2. *The left, right, and Riemann-Liouville fractional integrals of order* α *are given as follows* (*Kilbas et al., 2006)*

$$ {}_0 I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-n} u(\tau) d\,\tau, $$

$$ {}_t I_T^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{\alpha-n} u(\tau) d\,\tau $$

and

$$ {}_0 I_T^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^T |t-\tau|^{\alpha-n} u(\tau) d\,\tau. $$

Definition 3. *The main equation of the problem (1) is called Ulam-Hyers* σ*-type stable for some* $\sigma > 0$ *if there exists* $C_0 > 0$ *such that for each* $\epsilon > 0$ *and each* $v \in C([0,T], R)$ *satisfies the following inequality*

$$ \left| {}^{RC}_0 D_T^\alpha v(t) - f\big(t, v(t), v(\lambda t)\big) \right| \le \epsilon t^{-\sigma}, \qquad t \in (0,T), \tag{2} $$

where $v(0) = A, v(T) = B$, *there exists a solution* $u \in C([0,T], R)$ *of the problem (1) such that*

$$ |u(t) - v(t)| \le C_0 \epsilon, \qquad t \in [0,T]. $$

Remark 1. *The idea of Definition 3 was introduced in (Dien, 2024). If an equation is Ulam-Hyers* σ*-type stable then it is Ulam-Hyers stable in common sense as defined in (Sousa and Oliveira, 2018), however, the converse is not true.*

Next, we provide some properties concerning the Riemann-Liouville integrals and Riesz-Caputo fractional derivatives (Kilbas et al., 2006; Chen et al., 2017).

Lemma 1. *If* $u \in C^n(0,T)$, *then*

$$ {}_0 I_t^\alpha\, {}_0^C D_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k $$

and

$$ {}_t I_T^\alpha\, {}_t^C D_T^\alpha u(t) = (-1)^n \left[u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(T)}{k!} (T-t)^k \right]. $$

$$ {}_0I_T^{\alpha}\, {}^{RC}_{0}D_T^{\alpha} u(t) = \frac{1}{2}\left({}_0I_t^{\alpha}\, {}^{RC}_{0}D_t^{\alpha} + (-1)^n\ {}_tI_T^{\alpha}\, {}^{RC}_{t}D_T^{\alpha} \right) u(t). $$

In particularly, if $\alpha \in (0,1]$ *and* $u \in C(0,T)$, *then*

$$ {}_0I_T^{\alpha}\, {}^{RC}_{0}D_T^{\alpha} u(t) = u(t) - \frac{1}{2}(u(0) + u(T)). $$

Lemma 2 (Dien, 2021). *Let* $\gamma < \alpha \le 1$. *Then, we have*

$$ \int_0^t \tau^{-\gamma}(t-\tau)^{\alpha-1} d\tau = B(\alpha, 1-\gamma) t^{\alpha-\gamma}, $$

where $B(.,.)$ *is the Beta function.*

Lemma 3. *Let* $T > 0, \gamma < \alpha \le 1$. *Then, we have*

$$ \int_0^T \tau^{-\gamma}|t-\tau|^{\alpha-1} d\tau \le \left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma} \right) T^{\alpha-\gamma} $$

for any $t \in [0,T]$.

Proof. We have

$$ \int_0^T \tau^{-\gamma}|t-\tau|^{\alpha-1} d\tau \quad = \int_0^t \tau^{-\gamma}(t-\tau)^{\alpha-1} d\tau + \int_t^T \tau^{-\gamma}(\tau-t)^{\alpha-1} d\tau \tag{3} $$

For the first term, using Lemma 2, we have

$$ \int_0^t \tau^{-\gamma}(t-\tau)^{\alpha-1} d\tau = B(\alpha, 1-\gamma) t^{\alpha-\gamma} \le B(\alpha, 1-\gamma) T^{\alpha-\gamma} \tag{4} $$

For the second term, we divide into two cases:

In the first case: $T \le 2t$, we have

$$ \begin{aligned} \int_t^T \tau^{-\gamma}(\tau-t)^{\alpha-1} d\tau &\le t^{-\gamma} \int_t^{2t} (\tau-t)^{\alpha-1} d\tau \\ &= \frac{1}{\alpha} t^{\alpha-\gamma} \le \frac{1}{\alpha} T^{\alpha-\gamma} \end{aligned} \tag{5} $$

In the second case: $T > 2t$, then, we have

$$ \begin{aligned} \int_t^T \tau^{-\gamma}(\tau-t)^{\alpha-1} d\tau &= \int_t^{2t} \tau^{-\gamma}(\tau-t)^{\alpha-1} d\tau + \int_{2t}^T \tau^{-\gamma}(\tau-t)^{\alpha-1} d\tau \\ &\le t^{-\gamma} \int_t^{2t} (\tau-t)^{\alpha-1} d\tau + \int_{2t}^T \tau^{\alpha-\gamma-1} d\tau \\ &\le \frac{1}{\alpha} t^{\alpha-\gamma} + \frac{1}{\alpha-\gamma}\left(T^{\alpha-\gamma} - (2t)^{\alpha-\gamma}\right) \\ &\le \left(\frac{1}{\alpha} + \frac{1}{\alpha-\gamma} \right) T^{\alpha-\gamma} \end{aligned} \tag{6} $$

Combining (5) and (6), we get

$$\int_t^T \tau^{-\gamma}(\tau-t)^{\alpha-1}d\tau \le \left(\frac{1}{\alpha}+\frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma} \tag{7}$$

for all $t \in [0,T]$.

Pushing (4) and (7) into (3), we obtain

$$\int_0^T \tau^{-\gamma}|t-\tau|^{\alpha-1}d\tau \le \left(B(\alpha,1-\gamma)+\frac{1}{\alpha}+\frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}$$

.

This completes the proof of the lemma.

Lemma 4 (Principle contraction). *Let B be a Banach space. Suppose that $F: B \to B$ is a contraction mapping, i.e., there exists $k \in (0,1)$ such that*

$$|F(u)-F(v)| \le k|u-v| \text{ for all } u,v \in B.$$

Then F admits a unique fixed point in B, i.e., there is a unique $u_0 \in B$ such that $F(u_0)=u_0$.

3. Fundamental results

In this section, we state and prove the main results of the paper. We begin by posing an assumption for the source function.

Assumption (A1): There exist $\gamma < \alpha \le 1$ and $L > 0$ such that

$$|f(t,u_1,v_1)-f(t,u_2,v_2)| \le Lt^{-\gamma}(|u_1-v_1|+|u_2-v_2|),$$
$$|f(t,0,0)| \le Lt^{-\gamma}$$

for any $u_1,u_2,v_1,v_2 \in R$ and $t \in (0,T)$.

Based on the above assumption, we state the existence and uniqueness result as follows.

Theorem 1. *Let $\alpha \in (0,1], T > 0$. Suppose that Assumption (A1) is satisfied. If*

$$\frac{2L}{\Gamma(\alpha)}\left(B(\alpha,1-\gamma)+\frac{1}{\alpha}+\frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma} < 1,$$

then the problem (1) has a unique solution in $C[0,T]$.

Proof. Using Lemma 1, we have

$${}_0I_T^{\alpha} f\big(t,u(t),u(\lambda t)\big) = {}_0I_T^{\alpha}\,{}^{RC}_{\ 0}D_T^{\alpha} u(t) = u(t) - \frac{1}{2}\big(u(0)+u(T)\big)$$

or

$$u(t) = \frac{1}{2}(A+B) + \frac{1}{\Gamma(\alpha)}\int_0^T |\tau-t|^{\alpha-1} f(\tau,u(\tau),u(\lambda\tau))d\tau.$$

To continue, let us define the following operator $F: C[0,T] \to C[0,T]$ given by

$$Fu(t) = \frac{1}{2}(A+B) + \frac{1}{\Gamma(\alpha)}\int_0^T |\tau-t|^{\alpha-1} f\big(\tau,u(\tau),u(\lambda\tau)\big)d\tau. \tag{8}$$

Using Assumption (A1) and Lemma 3, for any $u,v \in C[0,T]$, we have

$$|Fu(t) - Fv(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^T |\tau - t|^{\alpha-1} |f(\tau, u(\tau), u(\lambda\tau)) - f(\tau, v(\tau), v(\lambda\tau))| d\tau$$

$$\le \frac{L}{\Gamma(\alpha)} \int_0^T \tau^{-\gamma} |\tau - t|^{\alpha-1} (|u(\tau) - v(\tau)| + |u(\lambda\tau) - v(\lambda\tau)|) d\tau.$$

Since $0 \le \lambda t \le t$, this implies that $|u(\lambda\tau) - v(\lambda\tau)| \le \|u - v\|$ for any $t \in [0, T]$. Therefore, we deduce from Lemma 3 that

$$|Fu(t) - Fv(t)| \le \frac{2L}{\Gamma(\alpha)} \|u - v\| \int_0^T \tau^{-\gamma} |\tau - t|^{\alpha-1} d\tau$$

$$\le \frac{2L}{\Gamma(\alpha)} \left(B(\alpha, 1 - \gamma) + \frac{1}{\alpha} + \frac{1}{\alpha - \gamma} \right) T^{\alpha-\gamma} \|u - v\|.$$

This leads to

$$||Fu - Fv| \le \frac{2L}{\Gamma(\alpha)} \left(B(\alpha, 1 - \gamma) + \frac{1}{\alpha} + \frac{1}{\alpha - \gamma} \right) T^{\alpha-\gamma} \|u - v\|.$$

Since $\frac{2L}{\Gamma(\alpha)} \left(B(\alpha, 1 - \gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma} \right) T^{\alpha-\gamma} < 1$, the latter inequality shows that F is a contraction mapping in $C[0, T]$. So, in view of Lemma 4, we conclude that F has a unique fixed point in $C[0, T]$, which is a solution to the problem (1). The proof of Theorem 1 is done.

Theorem 2. *Suppose that all the assumptions in Theorem 1 hold. Then the main equation of the problem (1) is Ulam-Hyers σ-type stable for any $\sigma \in [0, \alpha)$.*

Proof. According to Theorem 1, the problem (1) has a unique solution belongs to $C[0, T]$ satisfying

$$u(t) = Fu(t), \tag{9}$$

where F defined in (8). On the other hand, if v satisfies (2), then there exists $\varphi(t) \in C[0, T]$ and $|\varphi(t)| \le t^{-\sigma}$ such that

$${}_{0}^{RC}D_T^{\alpha} v(t) - f(t, v(t), v(\lambda t)) = \epsilon\varphi(t),$$

and $v(0) = A, v(T) = B$. This leads to

$$v(t) = \frac{1}{2}(A + B) + \frac{1}{\Gamma(\alpha)} \int_0^T |\tau - t|^{\alpha-1} (f(\tau, v(\tau), v(\lambda\tau)) + \epsilon\varphi(\tau)) d\tau$$

or

$$v(t) - Fv(t) = \frac{\epsilon}{\Gamma(\alpha)} \int_0^T |\tau - t|^{\alpha-1} \varphi(\tau) d\tau.$$

So, using Lemma 3, we get

$$|v(t) - Fv(t)| \le \frac{\epsilon}{\Gamma(\alpha)} \int_0^T |\tau - t|^{\alpha-1} \tau^{-\sigma} d\tau$$

$$\le \frac{\epsilon}{\Gamma(\alpha)} \left(B(\alpha, 1 - \sigma) + \frac{1}{\alpha} + \frac{1}{\alpha - \sigma} \right) T^{\alpha-\sigma}. \tag{10}$$

Furthermore, in the process of the proof of Theorem 1, we have proved that

$$|Fu(t) - Fv(t)| \le \frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}\|u-v\|.$$

Using the latter inequality together with (9) and (10), we obtain

$$\begin{aligned}|u(t) - v(t)| &= |Fu(t) - v(t)| \\ &\le |Fu(t) - Fv(t)| + |v(t) - Fv(t)| \\ &\le \frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}\|u-v\| \\ &\quad + \frac{\epsilon}{\Gamma(\alpha)}\left(B(\alpha, 1-\sigma) + \frac{1}{\alpha} + \frac{1}{\alpha-\sigma}\right)T^{\alpha-\sigma}.\end{aligned}$$

It implies that

$$\begin{aligned}\|u-v\| \le{}& \frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}\|u-v\| \\ &+ \frac{\epsilon}{\Gamma(\alpha)}\left(B(\alpha, 1-\sigma) + \frac{1}{\alpha} + \frac{1}{\alpha-\sigma}\right)T^{\alpha-\sigma}.\end{aligned}$$

This gives

$$\|u-v\| \le \frac{\frac{1}{\Gamma(\alpha)}\left(B(\alpha, 1-\sigma) + \frac{1}{\alpha} + \frac{1}{\alpha-\sigma}\right)T^{\alpha-\sigma}}{1 - \frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}}\epsilon$$

or

$$|u(t) - v(t)| \le \frac{\frac{1}{\Gamma(\alpha)}\left(B(\alpha, 1-\sigma) + \frac{1}{\alpha} + \frac{1}{\alpha-\sigma}\right)T^{\alpha-\sigma}}{1 - \frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma}}\epsilon$$

for all $t \in [0, T]$. This completes the proof of Theorem 2.

4. Examples

In this section, we construct two examples to validate the applicability of theoretical results.

Example 1. Consider the following problem

$$\begin{cases} {}^{RC}_{0}D^{0.9}_{T}u(t) = \frac{1}{20t^{0.6}}(\sin u(t) + u(0.8t)), \quad t \in (0,1), \\ u(0) = 0, u(T) = 0. \end{cases} \tag{11}$$

Herein, we have $T = 1, \alpha = 0.9, \gamma = 0.6, \lambda = 0.8$, and

$$\mathrm{f}(\mathrm{t}, \mathrm{u}(\mathrm{t}), \mathrm{u}(\lambda \mathrm{t})) = \frac{1}{20t^{0.6}}\left(\sin u(t) + u(0.8t)\right).$$

Using the fact that $|\sin u(t) - \sin v(t)| \le |u(t) - v(t)|$, we have

$$|\mathrm{f}(\mathrm{t}, \mathrm{u}(\mathrm{t}), \mathrm{u}(\lambda \mathrm{t})) - \mathrm{f}(\mathrm{t}, \mathrm{v}(\mathrm{t}), \mathrm{v}(\lambda \mathrm{t}))| = \frac{L}{t^{0.6}}(|u(t) - v(t)| + |u(0.8t) - v(0.8t)|),$$

where $L = 1/20$, and

$$\mathrm{f}(\mathrm{t}, 0, 0) = 0.$$

So, Assumption (A1) holds. On the other hand, we have

$$\frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma} \approx \frac{0.1}{1.07}(2.64 + 1.11 + 2,5) \approx 0.58 < 1.$$

We find that all the assumptions in Theorem 1 are satisfied. Therefore, using Theorem 1 and Theorem 1, we conclude that problem (11) has a unique solution and the main equation of the problem is Ulam-Hyers σ-stable for any $\sigma < 0.9$.

Example 2. Consider the following problem

$$\begin{cases} {}^{RC}_{0}D^{0.8}_{T}u(t) = \dfrac{1}{10t^{0.2}}\left(u(t) + \dfrac{1}{|u(0.9t)|+1} + 1\right), \quad t \in (0,1), \\ u(0) = 1, u(T) = 2. \end{cases} \tag{12}$$

We have $T = 1, \alpha = 0.8, \gamma = 0.2, \lambda = 0.9$, and

$$\mathrm{f}(\mathrm{t}, \mathrm{u}(\mathrm{t}), \mathrm{u}(\lambda \mathrm{t})) = \frac{1}{10t^{0.2}}\left(u(t) + \frac{1}{|u(0.9t)|+1} + 1\right).$$

By direct computation, we have

$$\left|\frac{1}{|u(0.9t)|+1} - \frac{1}{|v(0.9t)|+1}\right| \le |u(0.9t) - v(0.9t)|.$$

This leads to

$$|\mathrm{f}(\mathrm{t}, \mathrm{u}(\mathrm{t}), \mathrm{u}(\lambda\mathrm{t})) - \mathrm{f}(\mathrm{t}, \mathrm{v}(\mathrm{t}), \mathrm{v}(\lambda\mathrm{t}))| = \frac{L}{t^{0.2}}(|u(t) - v(t)| + |u(0.9t) - v(0.9t)|),$$

where $L = 1/10$, and

$$\mathrm{f}(\mathrm{t}, 0, 0) = \frac{1}{10t^{0.2}}.$$

It means that Assumption (A1) holds. On the other hand, we have

$$\frac{2L}{\Gamma(\alpha)}\left(B(\alpha, 1-\gamma) + \frac{1}{\alpha} + \frac{1}{\alpha-\gamma}\right)T^{\alpha-\gamma} \approx \frac{0.2}{1.16}(1.46 + 1.25 + 1.67) \approx 0.76 < 1.$$

So, all the assumptions in Theorem 1 hold. Using Theorem 1 and Theorem 1, we conclude that problem (12) has a unique solution and the main equation of the problem is Ulam-Hyers σ-stable for any $\sigma < 0.8$.

5. Conclusions

Fractional differential equations with weakly singular sources are harder to study than the other cases. To over this difficulty, we need to use some complex techniques. This paper has considered the pantograph differential equation connected to the Riesz-Caputo fractional derivative where the source function may have a singularity. Under some suitable conditions, we have shown that the problem has a unique solution. Moreover, we have also proved that the main equation of the problem is Ulam-Hyers σ-type stable. We have also constructed two examples to illustrate the obtained results. In future works, we would like to develop these results to general delay fractional equations with weakly singular sources.

References

Adjimi, N., Benbachir, M. (2022). Surveys in Existence results for Langevin equation with Riesz-Caputo fractional derivative. *Mathematics and its Applications*, Vol. 17, 225-239.

Agrawal, O. P. (2007). Fractional variational calculus in terms of Riesz fractional derivatives. *J. Phys. A: Math. Theor.* 40, 6287-6303.

Aleem, M., Rehman, M. U., Alzabut, J., Etemad, S., Rezapour,S (2021). On solutions of nonlinear BVPs with general boundary conditions by using a generalized Riesz–Caputo operator. *Advances in Difference Equations*, 303.

Almeida, R. (2012). Fractional variational problems with the Riesz-Caputo derivative. *Applied Mathematics Letters*, *25*(2), 142-148.

Chen, F., Baleanu, D., Wu, G.C. (2017). Existence results of fractional differential equations with Riesz-Caputo derivative. *Eur. Phys. J. Special Topics*, 226, 3411-3425.

Chen, F., Chen, A., Wu, X. (2019). Anti-periodic boundary value problems with Riesz–Caputo derivative. *Advances in Difference Equations*, 119.

Coffey, W. T., Kalmykov, Y. P., Waldron, J. T. (2004). *The Langevin equation, 2nd ed. World Scientific Series in Contemporary Chemical Physics 14*. World Scientific Publishing Co., River Edge, NJ.

Dien, N. M. (2021). Generalized weakly singular Gronwall-type inequalities and their applications to fractional differential equations. *Rocky Mountain J. Math. 51*(2), 689-707.

Dien, N. M. (2024). Some new uniqueness and Ulam-Hyes type stability results for nonlinear fractional neutral hybrid differential equations with time-varying lags. *Math. Slovaca 74*(2), 379-392

Doha, E. H., Bhrawy, A. H., Hafez, M. R. (2017). *Numerical algorithm for solving multi-pantograph delay equations on the half-line using Jacobi rational functions with convergence analysis*. Vol. 33, 297-310.

Goncalves, L.N., Fernandes, J., Ferraz, A., Silva, A. G., Sebastiao, P. J. (2020). Physical pendulum model: Fractional differential equation and memory effects. *Am. J. Phys. 88*(11), 962-975.

Gu, C. Y., Zhang, J., Wu, G.C. (2019). Positive solutions of fractional differential equations with the Riesz space derivative. *Applied Mathematics Letters*, 95, 59-64.

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). *Theory and applications of fractional differential equations. Elsevier*. Amsterdam, The Netherlands.

Iomin, A. (2019). Fractional evolution in quantum mechanics. *Chaos, Solitons & Fractals:* X(1), 100001.

Rahou, W., Salim, A., Lazreg, J. E., Benchohra, M. (2023). On Fractional Differential Equations with Riesz-Caputo Derivative and Non-Instantaneous Impulses. *Sahand Communications in Mathematical Analysis* (SCMA), *20*(3), 109-132.

Rihan, F.A. (2024). Continuous Runge–Kutta schemes for pantograph-type delay differential equations. *Partial Differential Equations in Applied Mathematics*, Vol. 11, 100797.

Sousa, J. V. da C. and Oliveira, E. C. de. (2018). On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. *J. Fixed Point Theory Appl. 20*(3), art. id. 96.

Tarasov, V. E. (2010). *Fractional dynamics*. Springer.

Toprakseven, S. (2023). Existence of solutions for fractional boundary value problems with Riesz space derivative and nonlocal conditions. Stud. Univ. *Babes-Bolyai Math. 68*(4), 701-715.