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On a nonlinear fractional differential equation with a weakly singular source

by ***Nguyen Minh Dien, Nguyen Thi Linh*** *(Thu Dau Mot university)*

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Corresponding author : *diennm@tdmu.edu.vn*
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ABSTRACT

This paper is devoted to the study of a nonlinear fractional differential equation with a weakly singular source in Banach space. Using Bielecki type norm, we show that the problem has a unique solution. Furthermore, we introduce a result of the new Ulam-Hyers type stability for the main equation.

Keywords: Fractional derivatives; Ulam-Hyers type stability; weakly singular source
2020 MSC: 34A08; 26A33; 34A12

1 Introduction

Fractional calculus has a long history, the first idea was proposed by Gottfried Leibniz in 1695. However, Riemann and Liouville were the first who build the foundation of fractional calculus. The first book published in 1974 by Oldham and Spanier (1974) is solely devoted to fractional calculus. They have observed that the derivatives/integrals of fractional orders are more useful than the integer derivatives/integrals. Nowadays, the area of fractional calculus has been fast developing and is presently being applied in all scientific fields. Fractional differential equations are an important branch of fractional calculus and have been studying in many works, we refer to (Diethehm, 2010; Kilbas et al, 2006) and references therein.

Recently, nonlinear fractional differential equations with weakly singular sources have been studied in various papers. Indeed, in (Delbosco and Rodino, 1996; Sin and Zheng, 2016), the authors considered fractional differential equations with weakly singular source, obtained some existence and uniqueness results for the corresponding problems. In (Dien and Viet, 2021; Dien, 2021) studied nonlinear fractional Langevin equations with time-singular coefficients, gave some results of the unique solution to the problem. Partial differential equations with weakly singular source also considered in some works (e.g., Dien, 2022). However, in the mentioned papers, to obtain the results of the unique solution for the problem, the authors have used some very complex techniques. In this work, we would like to introduce a new technique to solve some fractional differential equations with weakly singular sources in a simple way.

Along with the study of the existence and uniqueness of solutions, the study of the stability of differential equations is also of interest (see e.g. (Vanterler da C. Sousa and Capelas de Oliveira, 2018)). To enrich the theory of stability for fractional differential equations, this paper gives a new result of Ulam-Hyers-type stability for the main equation.

In the present paper, we consider a nonlinear fractional differential equation in a Banach space $(\mathbb{B}, \|\cdot\|)$ as follows

$$D_t^\alpha u(t) = f(t, u(t)), \quad t \in (0,T], \; n-1 < \alpha \le n, \; n \in \mathbb{N}^*, \tag{1.1}$$

subject to the conditions

$$u^{(k)}(0) = \xi_k, \tag{1.2}$$

where D_t^α is the Caputo fractional derivative of order α. Moreover, we examine the above problem under the assumption that the source function satisfies the following condition:

Hypothesis $(\mathcal{H}1)$***:*** *f is a singular Lipschitz function, i.e., there exist a positive number L_f and a non-negative number $\gamma < \alpha$ such that*

$$\|f(t,u(t)) - f(t,v(t))\| \le L_f t^{-\gamma} \|u - v\|,$$
$$\|f(t,0)\| \le L_f t^{-\gamma}$$

for all $t \in (0,T]$, $u, v \in \mathbb{B}$.

The new features of this work are: (i) introduce a new technique to show that the problem has a unique solution (ii) propose and study a new concept of Ulam-Hyers stability for the main equation.

The remainder of the paper is organized as follows. In section 2, we present some notations, definitions, and lemmas severing for the process of studying this work. The last section is devoted to stating and proving the main results of this study.

2 Preliminaries

In this section, we introduce some notations, definitions, and lemmas that we will use in the present paper.

Let use denote $\mathbb{B}$ the Banach space with the norm $\|\cdot\|$. For $u \in C([0,T],\mathbb{B})$, we denote

$$|||u||| = \sup_{0\le t\le T} \|u(t)\|.$$

We also recall the classical Gamma and Beta functions

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,\mathrm{d}t, \quad B(p,q) = \int_0^1 (1-t)^{p-1}t^{q-1}\,\mathrm{d}t, \quad p,q>0.$$

We continue by introducing the concepts of the fractional integral and Caputo fractional derivative.

Definition 2.1 (Kilbas et al, 2006)**.** *The Caputo derivative of fractional order $\alpha > 0$ is defined as*

$$D_t^\alpha h(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,\mathrm{d}s & as \;\; n-1<\alpha<n=[\alpha]+1, \\ h^{(n)}(t) & as \;\; \alpha = n, \end{cases}$$

where $[\alpha]$ denotes the integer part of the real number α.

Continuously, let us introduce a new definition of Ulam-Hyers type stability that we will study in this paper.

Definition 2.2. *Equation (1.1) is called Ulam-Hyers β-type stable if there exist $C>0$ and $0\le\omega<1$ such that for each $\epsilon>0$ and for each solution $v\in C([0,T],\mathbb{B})$ of the following inequality*

$$\|D_t^\alpha v(t) - f(t,v(t))\| \le \epsilon t^{-\omega} \quad t\in(0,T], \tag{2.1}$$

there exists a solution $u\in C([0,T],\mathbb{B})$ of Equation (1.1) such that

$$\|u(t)-v(t)\| \le C\epsilon, \quad t\in[0,T].$$

To end this section, we state and prove an inequality that plays an important role in proving the main results of the paper.

Lemma 2.3. *Let $T > 0$, $\alpha > 0$, $\gamma < \min\{1, \alpha\}$, and $\kappa > 1/T$. Then, we have*

$$\int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s \le C \frac{e^{\kappa t}}{\min\{\kappa^{1-\gamma},\ \kappa^{\alpha-\gamma}\}},$$

where C is a positive constant independent of κ.

Proof. If $t \in [0, 1/\kappa]$, we have

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s &\le e^{\kappa t} \int_0^t (t-s)^{\alpha-1} s^{-\gamma}\, \mathrm{d}s \\ &= e^{\kappa t} B(\alpha, 1-\gamma) t^{\alpha-\gamma} \le B(\alpha, 1-\gamma) e^{\kappa t}/\kappa^{\alpha-\gamma}. \end{aligned} \tag{2.2}$$

If $t \in (1/\kappa, T]$, we have

$$\int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s = I_1 + I_2, \tag{2.3}$$

where $I_1 = \displaystyle\int_0^{1/\kappa} (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s$, $\quad I_2 = \displaystyle\int_{1/\kappa}^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s$.
We now find some estimate for I_1 and I_2.
Estimate for I_1. If $\alpha < 1$, we have

$$\begin{aligned} I_1 &\le e^{\kappa t} \int_0^{1/\kappa} (1/\kappa - s)^{\alpha-1} s^{-\gamma}\, \mathrm{d}s \\ &= e^{\kappa t} B(\alpha, 1-\gamma) t^{\alpha-\gamma} \le e^{\kappa t} B(\alpha, 1-\gamma)/\kappa^{\alpha-\gamma}. \end{aligned} \tag{2.4}$$

If $\alpha \ge 1$, we have

$$\begin{aligned} I_1 &\le e^{\kappa t} T^{\alpha-1} \int_0^{1/\kappa} s^{-\gamma}\, \mathrm{d}s \\ &= e^{\kappa t} T^{\alpha-1} t^{1-\gamma} \le e^{\kappa t} T^{\alpha-1}/\kappa^{1-\gamma}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$I_1 \le C_1 e^{\kappa t} / \min\{\kappa^{1-\gamma},\ \kappa^{\alpha-\gamma}\}, \tag{2.6}$$

where $C_1 = \max\{B(\alpha, 1-\gamma),\, T^{\alpha-1}\}$.
Estimate for I_2.

$$I_2 \le \kappa^{\gamma} \int_{\kappa}^t (t-s)^{\alpha-1} e^{\kappa s}\, \mathrm{d}s$$

$$\begin{aligned}
&= \kappa^{\gamma} e^{\kappa t} \int_0^{t-1/\kappa} s^{\alpha-1} e^{-\kappa s}\, \mathrm{d}s \\
&= \kappa^{\gamma-\alpha} e^{\kappa t} \int_0^{\kappa(t-1/\kappa)} s^{\alpha-1} e^{-s}\, \mathrm{d}s \le \Gamma(\alpha) e^{\kappa t}/\kappa^{\alpha-\gamma}.
\end{aligned} \tag{2.7}$$

Pushing (2.6) and (2.7) into (2.3), we obtain

$$\int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s}\, \mathrm{d}s \le C_2 \frac{e^{\kappa t}}{\min\{\kappa^{1-\gamma},\ \kappa^{\alpha-\gamma}\}},$$

where $C_2 = \max\{C_1, \Gamma(\alpha)\}$. Combining the latter inequality with (2.2), we obtain the desired result of Lemma. □

3 Fundamental results

This section is devoted to presenting the main results of the paper. As known in (Dien, 22), problem (1.1)-(1.2) can be transformed to the following integral equation.

$$u(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, \mathrm{d}s, \tag{3.1}$$

where $\vartheta(t) = \sum_{k=0}^{n-1} \xi_k \frac{t^k}{k!}$.

From now on, let us consider the Bielecki type norm as follows

$$\|\!|u|\!\|_{\kappa} = \max_{0\le t\le T} e^{-\kappa t} \|u(t)\|.$$

Using the integral just obtained, we state and prove the main results of the paper. The first is the result of the existence and uniqueness of solutions to the problem.

Theorem 3.1. *If Hypothesis $(\mathcal{H}1)$ is satisfied, then problem (1.1)-(1.2) has a unique solution in $C([0,T],\mathbb{B})$.*

PROOF. We consider the operator $Q : C([0,T],\mathbb{B}) \to C([0,T],\mathbb{B})$ defined as follows

$$Qu(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, \mathrm{d}s, \tag{3.2}$$

where $\vartheta(t) = \sum_{k=0}^{n-1} \xi_k \frac{t^k}{k!}$. We will show that Q is contraction mapping when κ sufficient large. In fact, for any $u, v \in C([0,T], \mathbb{B})$ and $\kappa > 1/T$, we have

$$\begin{aligned}
\|Qu(t) - Qv(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s)) - f(s,v(s))\| \, \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} \|u(s) - v(s)\| \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s} e^{-\kappa s} \|u(s) - v(s)\| \, \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} |||u - v|||_\kappa \int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s} \, \mathrm{d}s.
\end{aligned}$$

Applying Lemma 2.3, we obtain

$$\|Qu(t) - Qv(t)\| \le \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} |||u - v|||_\kappa e^{\kappa t},$$

where C is a constant independent of κ. The latter inequality yields

$$|||Qu - Qv|||_\kappa \le \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} |||u - v|||_\kappa.$$

Since $\frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} \to 0$ as $\kappa \to +\infty$, this leads to from the latter inequality that Q is contraction when κ sufficient large. In this case, Q admits a unique fixed point in $C([0,T], \mathbb{B})$, which is a (unique) solution of problem (1.1)-(1.2). This completes the proof of Theorem. □

Remark 3.2. *To the best of our knowledge, this seems the first time in the literature that the result of the unique solution has been proved by using the Bielecki-type norm. Our method is shortened and more convenient than the previous one.*

Next, we give a result on Ulam-Hyers β-type stable for the main equation.

Theorem 3.3. *Suppose that Hypothesis* $(\mathcal{H}1)$ *is valid. Then, the main equation (1.1) is Ulam-Hyers* β*-type stable for some* $\beta < \min\{1, \alpha\}$.

Remark 3.4. *The Ulam-Hyers* ω*-type stable leads to the Ulam-Hyers stable in common sense (see e.g. (Vanterler da C. Sousa and Capelas de Oliveira, 2018)), but the converse is not true. So, our result is stronger than the previous one.*

PROOF. Since Hypothesis ($\mathcal{H}1$) is valid, problem (1.1)-(1.2) has a unique solution and it satisfies the integral (3.1). We continue by consider a solution v of the following inequality

$$\|D_t^\alpha v(t) - f(t, v(t))\| \leq \epsilon t^{-\beta}, \quad t \in (0, T].$$

The latter inequality shows that there exist a non-negative function $\varphi \in C([0,T], \mathbb{B})$ such that $\|\varphi(t)\| \leq \epsilon t^{-\beta}$ for any $t \in (0, T]$ and

$$D_t^\alpha v(t) = f(t, v(t)) + \varphi(t), \quad t \in (0, T].$$

The latter equation yields

$$v(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, v(s)) + \varphi(t)) \, \mathrm{d}s,$$

where $\vartheta(t) = \sum_{k=0}^{n-1} \xi_k \frac{t^k}{k!}$. It follows that

$$\begin{aligned}
\|v(t) - u(t)\| &= \left\| v(t) - \vartheta(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) \, \mathrm{d}s \right\| \\
&\leq \left\| v(t) - \vartheta(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s)) \, \mathrm{d}s \right\| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, v(s)) - f(s, u(s))\| \, \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\varphi(s)\| \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} \|v(s) - u(s)\| \, \mathrm{d}s \\
&\leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\beta} \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} |||v - u|||_\kappa \int_0^t (t-s)^{\alpha-1} s^{-\gamma} e^{\kappa s} \, \mathrm{d}s \\
&\leq \frac{\epsilon}{\Gamma(\alpha)} B(\alpha, 1-\beta) t^{\alpha-\beta} + \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} |||v - u|||_\kappa e^{\kappa t} \\
&\leq \frac{\epsilon}{\Gamma(\alpha)} B(\alpha, 1-\beta) T^{\alpha-\beta} + \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} |||v - u|||_\kappa e^{\kappa t}.
\end{aligned}$$

From the inequality just obtained deduces

$$|||v - u|||_\kappa \leq \frac{\epsilon}{\Gamma(\alpha)} B(\alpha, 1-\beta) T^{\alpha-\beta} + \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} |||v - u|||_\kappa.$$

Choosing κ sufficient large such that $K = \frac{C}{\Gamma(\alpha) \min\{\kappa^{1-\gamma}, \ \kappa^{\alpha-\gamma}\}} < 1$, then, we have

$$|||v - u|||_\kappa \leq \frac{1}{1-K} \frac{\epsilon}{\Gamma(\alpha)} B(\alpha, 1-\beta) T^{\alpha-\beta}.$$

It follows

$$\|v(t)-u(t)\| \leq \frac{1}{1-K}\frac{\epsilon}{\Gamma(\alpha)}B(\alpha,1-\beta)T^{\alpha-\beta}e^{\kappa t} \leq \frac{1}{1-K}\frac{\epsilon}{\Gamma(\alpha)}B(\alpha,1-\beta)T^{\alpha-\beta}e^{\kappa T}$$

The proof of Theorem is done. □

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