LYAPUNOV-TYPE INEQUALITY FOR BOUNDARY VALUE PROBLEMS WITH HADAMARD FRACTIONAL DERIVATIVES AND APPLICATIONS

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Abstract

In this paper, we consider a boundary value problem involving the Hadamard fractional derivative. We establish a Lyapunov-type inequality for the problem by constructing the green function and analyzing its properties. Next, we employ a fixed-point theorem to obtain the existence and uniqueness of the solution to the problem. The paper concludes with three examples that illustrate the theoretical results.

Keywords: Fixed point theorems, Hadamard fractional derivative, Lyapunov-type inequalities.

1. Introduction

1.1. Background and literature

Recently, boundary value problems involving fractional-order derivatives have attracted considerable attention and extensive research due to their wide range of applications in various fields such as physics, mechanics, biology, engineering, signal processing, etc. "Investigating the properties of solutions, including existence, eigenvalue estimates, and Lyapunov-type inequalities, plays a crucial role". Therefore, investigating the properties of boundary value problems involving different types of fractional derivatives is significant both in theory and in practical applications. Fractional derivatives and several related problems are introduced in the book (Kibas et al, 2006; Miller and Ross, 1993).

Boundary value problems involving Hadamard fractional derivatives have also been presented in (Benhamida et al, 2018; He et al 2022). However, in these studies, the order of the derivative α is relatively small (typically $\alpha \in (1,2]$). Hence, extending the research to problems with more general fractional orders is necessary.

A Lyapunov-type inequality was investigated in (Dien, 2021) and by several other authors mentioned in (Ntouyas et al, 2022). However, the inequality for the problem with general fractional boundary conditions involving the Hadamard derivative has not yet been thoroughly studied.

1.2. The aim of the paper

Motivated by the above reasons, in this paper, we establish a Lyapunov-type inequality for a general fractional boundary value problem involving the Hadamard fractional derivative of arbitrary order. More specifically, with $n \geq 2, n-1 < \alpha \leq n$ và $n-2 \leq \gamma \leq n-1$, let $\alpha - n + k \leq \alpha_k < \alpha - n + k + 1$ for $k = 1,2,..,n-2$, we consider the following problem:

$$\begin{cases} {}^{H}D^{\alpha}u(t) = f\big(t, u(t)\big), 1 < t < b, \\ u(1) = {}^{H}D^{\alpha_1}u(1) =..= {}^{H}D^{\alpha_{n-2}}u(1) = {}^{H}D^{\gamma}u(b) = 0. \end{cases} \quad (1.1)$$

We construct the Green's function for problem (1.1), derive an upper bound estimate for this Green's function, establish a Lyapunov-type inequality, and present a result on the existence and uniqueness of the solution to problem (1.1).

2. Preliminaries

In this section, we present the notations, definitions, and preliminary concepts that will be used throughout the remainder of the paper.

We begin by introducing the concepts and some properties related to the Hadamard fractional integral and derivative. These concepts and properties have been presented in various references, such as (Benhamida et al, 2018; He et al 2022).

Definition 2.1. *The Hadamard fractional integral of order* α *of a function:* $[1, +\infty) \to \mathbb{R}$ *is defined by*

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} \mathrm{d}s, \qquad \alpha > 0.$$

Definition 2.2. *The Hadamard fractional derivative of order* α *of a function* h *on the interval* $[1, +\infty)$ *is defined by*

$${}^{H}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s} \mathrm{d}s, \qquad n-1 < \alpha < n.$$

Lemma 2.3**.** *Let* $\alpha \geq 0$. *Then, the differential equation*

$${}^{H}D^{\alpha}h(t) = 0,$$

has the general solution

$$h(t) = \sum_{j=1}^{n} c_j (\ln t)^{\alpha - j},$$

moreover

$$I^{\alpha H}D^{\alpha}h(t) = h(t) + \sum_{j=1}^{n} c_j (\ln t)^{\alpha - j}.$$

Lemma 2.4. *If* $\alpha > 0, \beta > 0$ *and* $0 < a < \infty$, *then*

$$I^{\beta}\left(\ln\frac{t}{a}\right)^{\alpha-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\left(\ln\frac{x}{a}\right)^{\alpha+\beta-1}$$

and

$${}^{H}D^{\beta}\left(\ln\frac{t}{a}\right)^{\alpha-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\left(\ln\frac{x}{a}\right)^{\alpha-\beta-1}.$$

Next, we introduce the contraction mapping principle in a Banach space. This is a well-known result that has been presented in many references, such as (Zeidler, 1986).

Proposition 2.5. *(Banach's contraction mapping principle). Let* $(X, \|\cdot\|_X)$ *be a Banach space and let* Ω *be a closed subset of* X. *Suppose that* $P: \Omega \to \Omega$ *satisfies the condition,* $k \in (0,1)$

$$\|P(u) - P(v)\|_X \leq k\|u - v\|_X, \qquad \forall u, v \in \Omega. \tag{2.1}$$

Then, there exists a unique fixed point of P *in* Ω, *that is, there exists a unique element* $u_0 \in \Omega$ *such that* $P(u_0) = u_0$.

3. Main results

3.1. Green's function

In this section, we introduce the Green's function of problem (1.1). In addition, we provide an upper bound estimate for this Green's function.

Proposition 3.1. *If u is a solution of problem (1.1), then u satisfies the following equation:*

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} G(t,\tau) f\big(\tau, u(\tau)\big)\mathrm{d}\tau,$$

where

$$G(t,\tau) = \begin{cases} \dfrac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}} - \dfrac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1}}, & 1 \leq \tau \leq t \leq b, \\ \dfrac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}}, & 1 \leq t \leq \tau \leq b. \end{cases}$$

Proof. We use Lemma (2.3), the general solution of equation (1.1) has the form

$$u(t) = I^{\alpha} f(t, u(t)) + \sum_{k=0}^{n-1} c_k (\ln t)^{\alpha-n+k}. \tag{3.1}$$

At $t=1$, we obtain $c_0=0$. Replacing $c_0=0$ in (3.1), we have

$$ {}^{H}D^{\alpha_p}u(t)=I^{\alpha-\alpha_p}f(t,u(t))+\sum_{k=p}^{n-1}c_k\frac{\Gamma(\alpha-n+k+1)}{\Gamma(\alpha-n+k+1-\alpha_p)}(\ln t)^{\alpha-n+k-\alpha_p}. $$

By successively applying the condition ${}^{H}D^{\alpha_p}u(1)=0$ for $p=1,2,..,n-2$, we obtain $c_p=0$ for all $p=1,2,..,n-2$. Finally, we also have

$$ {}^{H}D^{\gamma}u(t)=I^{\alpha-\gamma}f(t,u(t))+c_{n-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}(\ln t)^{\alpha-\gamma-1}. $$

Since ${}^{H}D^{\gamma}_{a+}u(b)=0$, it is easy to see that

$$ \begin{aligned} c_{n-1}&=-\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)(\ln b)^{\alpha-\gamma-1}}I^{\alpha-\gamma}f\big(t,u(t)\big)\Big|_{t=b}\\ &=-\frac{1}{\Gamma(\alpha)}\int_1^b\frac{1}{\tau}\left(\frac{\ln\frac{b}{\tau}}{\ln b}\right)^{\alpha-\gamma-1}f\big(\tau,u(\tau)\big)\mathrm{d}\tau. \end{aligned} $$

Substituting all the coefficients c_k for $k=1,2,..,n-1$ into (3.1), we obtain

$$ \begin{aligned} u(t)=&\frac{1}{\Gamma(\alpha)}\int_1^t\frac{1}{\tau}\left(\ln\frac{t}{\tau}\right)^{\alpha-1}f\big(\tau,u(\tau)\big)\mathrm{d}\tau\\ &-\frac{1}{\Gamma(\alpha)}\int_1^b\frac{1}{\tau}\left(\frac{\ln\frac{b}{\tau}}{\ln b}\right)^{\alpha-\gamma-1}(\ln t)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau\\ =&\frac{1}{\Gamma(\alpha)}\int_1^t\frac{1}{\tau}\left(\ln\frac{t}{\tau}\right)^{\alpha-1}f\big(\tau,u(\tau)\big)\mathrm{d}\tau\\ &-\frac{1}{\Gamma(\alpha)}\int_1^t\frac{1}{\tau}\left(\frac{\ln\frac{t}{\tau}}{\ln b}\right)^{\alpha-\gamma-1}(\ln t)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau\\ &-\frac{1}{\Gamma(\alpha)}\int_t^b\frac{1}{\tau}\left(\frac{\ln\frac{b}{\tau}}{\ln b}\right)^{\alpha-\gamma-1}(\ln t)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau\\ =&\frac{1}{\Gamma(\alpha)}\int_1^b\frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1}G(t,\tau)f\big(\tau,u(\tau)\big)\mathrm{d}\tau, \end{aligned} $$

where

$$ G(t,\tau)=\begin{cases}\dfrac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}}-\dfrac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1}}, & 1\le\tau\le t\le b,\\[2ex] \dfrac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}}, & 1\le t\le\tau\le b.\end{cases} $$

□

Definition 3.2. *The function G in Proposition 3.1 is called the green function of the problem (1.1).*

Remark 3.3. *The Green's function associated with problem (1.1) differs from the one defined in* (Dhar and Neugebauer, 2022). *This type of Green's function was investigated in* (Dien and Nieto, 2022) *in the context of nonlinear continuous fractional boundary value problems.*

Proposition 3.4. *The Green's function given in Proposition 3.1 satisfies the following properties:*
(i) $G(t,\tau) \geq 0$,
(ii) $G_\tau(t,\tau) \geq 0 (1 \leq \tau \leq t \leq b)$. *It follows that*

$$\max_{1 \leq \tau \leq t \leq b} G(t,\tau) = \max_{t \in [1,b]} \frac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}} = (\ln b)^{\gamma}.$$

Proof. For $1 \leq t \leq \tau \leq b$, (i) is obviously true.
for $1 \leq \tau \leq t \leq b$, We study the function

$$G(t,\tau) = \frac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}} - \frac{\left(\ln \frac{t}{\tau}\right)^{\alpha-1}}{\left(\ln \frac{b}{\tau}\right)^{\alpha-\gamma-1}},$$

By direct computation, we obtain

$$G_\tau(t,\tau) = \frac{\left(\ln \frac{t}{\tau}\right)^{\alpha-1}}{\tau \left(\ln \frac{b}{\tau}\right)^{\alpha-\gamma-1}} \left(\frac{\alpha-1}{\ln \frac{t}{\tau}} - \frac{\alpha-\gamma-1}{\ln \frac{b}{\tau}} \right),$$

Since $\gamma \geq 0$ and $1 \leq \tau \leq t \leq b$, we have $\alpha - 1 \geq \alpha - \gamma - 1$ và $\ln \frac{b}{\tau} \geq \ln \frac{t}{\tau} \geq 0$. It follows that $G_\tau(t,\tau) \geq 0$. From the monotonicity of the function, we conclude that $G(t,\tau)$ is increasing with respect to τ hence

$$G(t,\tau) \geq G(t,1) = 0$$

and

$$G(t,\tau) \leq G(t,t) = \frac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}}.$$

It follows that

$$\max_{1 \leq \tau \leq t \leq b} G(t,\tau) = \max_{t \in [1,b]} \frac{(\ln t)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}} = \frac{(\ln b)^{\alpha-1}}{(\ln b)^{\alpha-\gamma-1}} = (\ln b)^{\gamma}.$$

The proof of Proposition is completed.

□

3.2. Lyapunov-type inequality

In this section, we introduce a Lyapunov-type inequality for problem (1.1).

Theorem 3.5. *Assume that there exists a function* $q:[1,b]\to\mathbb{R}_+$ *such that*

$$|f(t,u)| = q(t)|u|,$$

for all $t\in(1,b]$. *Let* $p(t)=\frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1}q(t)$. *If* $p(.)\in L^1(1,b)$ *and problem (1.1) has a nontrivial solution, then*

$$\int_1^b \frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1} q(t)dt \geq \frac{\Gamma(\alpha)}{(\ln b)^\gamma}.$$

Proof. From Proposition 3.1, if problem (1.1) has a nontrivial solution, then

$$\begin{aligned} u(t) &\leq \frac{1}{\Gamma(\alpha)}\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} |G(t,\tau)|\,|f(\tau,u(\tau)|d\tau \\ &\leq \frac{(\ln b)^\gamma}{\Gamma(\alpha)}\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} q(\tau)|u(\tau)|d\tau \\ &\leq \frac{(\ln b)^\gamma}{\Gamma(\alpha)}\|u\|\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} q(\tau)d\tau. \end{aligned}$$

The inequality just obtained implies that

$$\|u\| \leq \frac{(\ln b)^\gamma}{\Gamma(\alpha)}\|u\|\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} q(\tau)d\tau.$$

Hence, we obtain

$$\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} q(\tau)\mathrm{d}\tau = \int_1^b \frac{1}{t}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} q(t)\mathrm{d}t \geq \frac{\Gamma(\alpha)}{(\ln b)^\gamma}.$$

This completes the proof of Theorem 3.5.

□

We have the following consequences of the Lyapunov-type inequality:

Corollary 3.6. *Assume that there exists a function* $q:(1,b]\to\mathbb{R}_+$ *such that*

$$|f(t,u)| \leq q(t)|u|,$$

for all $t\in(1,b]$. *Let* $p(t)=\frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1}q(t)$. *If* $p(\cdot)\in L^1(1,b)$ *and inequality*

$$\int_1^b \frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1} q(t)\mathrm{d}t < \frac{\Gamma(\alpha)}{(\ln b)^\gamma},$$

is satisfied, then problem (1.1) admits no nontrivial solution.

Proof. Assume that the problem has a nontrivial solution. Then, by Theorem 3.5, we have

$$\int_1^b \frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1} q(t)\mathrm{d}t \geq \frac{\Gamma(\alpha)}{(\ln b)^\gamma}.$$

This contradicts the assumption. Therefore, the problem has no nontrivial solution.

□

Corollary 3.7. *Assume that λ is an eigenvalue of the problem*

$$\begin{cases} {}^{H}D^{\alpha}u(t) = \lambda u, 1 < t < b, \\ u(1) = {}^{H}D^{\alpha_1}u(1) =..= {}^{H}D^{\alpha_{n-2}}u(1) = {}^{H}D^{\gamma}u(b) = 0, \end{cases}$$

then

$$|\lambda| \geq \frac{(\alpha-\gamma)\Gamma(\alpha)}{(\ln b)^{\alpha}}.$$

Proof. λ is an eigenvalue, which means that the problem has a nontrivial solution u, applying Theorem 3.5 for $q(t) = |\lambda|$, we obtain

$$\int_1^b \frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1} |\lambda| \mathrm{d}t \geq \frac{\Gamma(\alpha)}{(\ln b)^{\gamma}},$$

hence

$$\frac{1}{\alpha-\gamma}(\ln b)^{\alpha-\gamma}|\lambda| \geq \frac{\Gamma(\alpha)}{(\ln b)^{\gamma}}.$$

The final inequality implies

$$|\lambda| \geq \frac{(\alpha-\gamma)\Gamma(\alpha)}{(\ln b)^{\alpha}}.$$

This completes the proof of the corollary.

□

3.3. Existence and uniqueness result via Banach's fixed point theorem

In this section, we present a result on the existence and uniqueness of the solution to problem (1.1). More precisely, we have the following theorem:

Theorem 3.8. *Assume that f is a Lipschitz continuous function, that is, there exists a constant $K > 0$ such that*

$$|f(t,u) - f(t,v)| \leq K|u-v|, \tag{3.2}$$

for all $t \in [1,b]$ và $u, v \in \mathbb{R}$. If

$$\frac{K(\ln b)^{\alpha}}{\Gamma(\alpha)(\alpha-\gamma)} < 1,$$

holds, then problem (1.1) has a unique solution in $C[1,b]$.

Proof. Let $\mathbb{B}$ be the space of continuous functions on $[1,b]$ equipped with the norm

$$\|u\| = \max_{t\in[1,b]} |u(t)|.$$

If $u(t)$ is a solution of problem (1.1), then $u(t)$ satisfies the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} G(t,\tau) f\big(\tau, u(\tau)\big)\mathrm{d}\tau.$$

We define the operator $T: \mathbb{B} \to \mathbb{B}$ is defined by

$$Tu(t) = \frac{1}{\Gamma(\alpha)}\int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} G(t,\tau) f\big(\tau, u(\tau)\big)\mathrm{d}\tau.$$

Then T is completely continuous. We show that T has a unique fixed point in $\mathbb{B}$. Indeed, for all $u, v \in \mathbb{B}$, we have

$$|Tu(t) - Tv(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} |G(t,\tau)|\left|f\big(\tau,u(\tau)\big) - f\big(\tau,v(\tau)\big)\right| \mathrm{d}\tau.$$

Given that $G(t,\tau) \geq 0$ trên $[1,b] \times [1,b]$, and by applying condition (3.2), it follows that

$$\begin{aligned}|Tu(t) - Tv(t)| &\leq \frac{K}{\Gamma(\alpha)} \int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} G(t,\tau)|u(\tau) - v(\tau)| d\tau \\ &\leq \frac{K}{\Gamma(\alpha)} \|u - v\| \int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} G(t,\tau) d\tau.\end{aligned}$$

Applying Proposition 3.4, we obtain

$$\begin{aligned}|Tu(t) - Tv(t)| &\leq \frac{K}{\Gamma(\alpha)} \|u - v\| (\ln b)^{\gamma} \int_1^b \frac{1}{\tau}\left(\ln\frac{b}{\tau}\right)^{\alpha-\gamma-1} d\tau \\ &= \frac{K}{\Gamma(\alpha)} \|u - v\| (\ln b)^{\gamma} \frac{(\ln b)^{\alpha-\gamma}}{\alpha - \gamma} \\ &= \frac{K(\ln\ b)^{\alpha}}{\Gamma(\alpha)(\alpha - \gamma)} \|u - v\|.\end{aligned}$$

This inequality implies

$$|Tu(t) - \mathrm{T}v(t)| \leq \frac{K(\ln b)^{\alpha}}{\Gamma(\alpha)(\alpha - \gamma)} \|u - v\|.$$

Since $\frac{K(\ln b)^{\alpha}}{\Gamma(\alpha)(\alpha-\gamma)} < 1$, the last inequality shows that T is a contraction mapping in the space $C[1,b]$. Therefore, T has a unique fixed point, which implies that problem (1.1) has a unique solution in $C[1,b]$.

□

4. Examples

In this section, we present some concrete examples to illustrate the theoretical results obtained above.

Example 4.1. *Prove that problem*

$$\begin{cases} {}^{H}D^{2.9}u(t) = 3.2(\ln t)^{-0.5} \sin u(t), & 1 < t < 2, \\ u(1) = {}^{H}D^{1.1}u(1) = {}^{H}D^{1.5}u(2) = 0, \end{cases}$$

admits a unique solution $u(t) = 0$ *for all* $t \in [1,2]$.

Proof. Let $b = 2, \alpha = 2.9, \gamma_1 = 1.1, \gamma = 1.5$ and $f(t,u) = 3.2(\ln t)^{-0.5}\sin\ u(t)$. Clearly, $u(t) = 0$ for all $t \in [1,2]$ is a solution of the given problem. We will show that this problem admits only the above unique solution. Using the inequality $|\sin x| \leq |u|$ for all u, we obtain

$$|f(t,u)| \leq q(t)|u|,$$

where

$$q(t) = 3.2(\ln t)^{-0.5}, \qquad t \in (1,2].$$

Indeed, we have

$$\begin{aligned} I &= \int_1^b \frac{1}{t}\left(\ln\frac{b}{t}\right)^{\alpha-\gamma-1} q(t)\mathrm{d}t \\ &= 3.2\int_1^2 \frac{1}{t}\left(\ln\frac{2}{t}\right)^{0.4} (\ln t)^{-0.5}\,\mathrm{d}t \\ &= 3.2\int_1^2 (\ln 2 - \ln t)^{0.4}(\ln t)^{-0.5}\,\mathrm{d}(\ln t). \end{aligned}$$

Let $v = \ln t$, so that $\mathrm{d}v = \frac{1}{t}\,\mathrm{d}t$, Substituting this change of variable into the integral, we obtain

$$I = 3.2\int_0^{\ln 2} (\ln 2 - v)^{0.4} v^{-0.5}\,\mathrm{d}v.$$

Next, let $v = x\ln 2$, so that $\mathrm{d}v = \ln 2\,\mathrm{d}x$, Substituting this change of variable into the integral, we obtain

$$\begin{aligned} I &= 3.2\int_0^1 (\ln 2 - x\ln 2)^{0.4}(x\ln 2)^{-0.5}(\ln 2)\mathrm{d}x \\ &= 3.2(\ln 2)^{1+0.4-0.5}\int_0^1 (1-x)^{0.4}x^{-0.5}\,\mathrm{d}x \\ &= 3.2(\ln 2)^{0.9}B(1.4,0.5) \\ &= 3.2(\ln 2)^{0.9}\frac{\Gamma(1.4)\Gamma(0.5)}{\Gamma(1.9)} \\ &\approx 3.83. \end{aligned}$$

Moreover, we also have

$$\frac{\Gamma(\alpha)}{(\ln b)^{\gamma}} = \frac{\Gamma(2.9)}{(\ln 2)^{1.5}} \approx 3.916.$$

Hence, we obtain

$$\int_1^b (b-t)^{\alpha-\gamma-1}q(t)\mathrm{d}t \approx 3.83 < 3.916 \approx \frac{\Gamma(\alpha)}{(\ln b)^{\gamma}}.$$

The inequality above confirms that the assumption in Corollary 3.6 holds. It follows that the given problem has the unique solution $u(t) = 0$ for all $t \in [1,2]$.

□

Example 4.2. *Show that the absolute value of every eigenvalue of problem*

$$\begin{cases} {}^{H}D^{1.7}u(t) = \lambda u(t), & 1 < t < b, \\ u(1) = {}^{H}D^{0.5}u(2) = 0, \end{cases}$$

is greater than 2.

Proof. We have $b = 2, \alpha = 1.7, \gamma = 0.5$. By Corollary 3.7, we have

$$\begin{aligned} |\lambda| &\geq \frac{(\alpha-\gamma)\Gamma(\alpha)}{(\ln b)^{\alpha}} \\ &\geq \frac{(1.7-0.5)\Gamma(1.7)}{(\ln 2)^{1.7}} \\ &\approx 2.014 > 2. \end{aligned}$$

This completes the proof.

□

Example 4.3. *Prove that problem*

$$\begin{cases} {}^{H}D^{1.5}u(t) = \dfrac{0.3tu(t)}{1+u^2(t)}, & t \in (1, e), \\ u(1) = {}^{H}D^{0.3}u(e) = 0, \end{cases}$$

has a unique solution on $C[1, e]$.

Proof. Let $f(t,u) = \frac{0.3tu}{1+u^2}$, for all $u, v \in \mathbb{R}$, we have

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left|\frac{0.3tu}{1+u^2} - \frac{0.3tv}{1+v^2}\right| \\ &= 0.3t|u-v| \left|\frac{1-uv}{(1+u^2)(1+v^2)}\right| \\ &\leq 0.3t|u-v| \left|\frac{1+\dfrac{u^2+v^2}{2}}{(1+u^2)(1+v^2)}\right| \\ &= 0.3t|u-v| \left|\frac{1}{2(1+u^2)} + \frac{1}{2(1+v^2)}\right| \\ &\leq 0.3t|u-v|. \end{aligned}$$

Since $0.3t \leq 0.3e < 1$, the function $f(t,u)$ satisfies the Lipschitz condition with respect to u with Lipschitz constant $K = 0.3e$. On the other hand, for $\alpha = 1.5, \gamma = 0.3, b = e$, we have

$$\frac{K(\ln b)^{\alpha}}{\Gamma(\alpha)(\alpha-\gamma)} = \frac{0.3e(\ln e)^{1.5}}{\Gamma(1.5)(1.5-1.3)} \approx 0{,}767 < 1.$$

The above inequality shows that the condition in Theorem 3.8 is satisfied. Consequently, we conclude that the given problem has a unique solution.

□

5. Concluding remarks

Fractional differential equations have recently attracted significant research interest. In this paper, we have established a Lyapunov-type inequality for a boundary value problem involving the Hadamard fractional derivative and obtained results concerning the existence and uniqueness of its solution. In future work, we aim to investigate Lyapunov-type inequalities for differential equations involving other types of fractional derivatives, such as the Caputo and Grunwald-Letnikov derivatives.

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