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Initial value problems for nonlinear fractional equations with discontinuous sources

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ABSTRACT

In this paper, we consider initial value problems for nonlinear fractional equations where source functions may discontinuous. We obtain the existence and uniqueness of maximal mild solutions of the problem. We also give some appropriate conditions such that mild solutions of the problem blow-up at a finite time. Furthermore, we discuss the continuous dependence of mild solutions of the problem with respect to fractional order.

Keywords: *Caputo fractional derivative, Initial value problems, Locally Lipschitz functions, Fixed point theorems*

1 Introduction

Fractional differential equations is a generalized form of an integer order differential equations where derivatives of integer orders are replaced by derivatives of arbitrary orders. There are numerous papers studying fractional differential equations with various boundary conditions. For initial value problems, we refer to the excellent monograph books (Diethelm, 2010; Kilbas et al, 2006). In this paper, we would like to enrich the theory of fractional differential equations by considering the following problem

$$\begin{cases} D_t^\alpha u(t) = f(t,u(t)), & t \in (0,T],\ n-1 < \alpha \le n \text{ for some } n \in \mathbb{N}^*, \\ u^{(k)}(0) = \xi_k, & (k = 0,1,2,..,n-1), \end{cases} \tag{1.1}$$

where $T > 0$ is given, and D_t^α is the Caputo fractional derivative of order α.

Unlike most of previous works, in this paper, the source function of the problem is assumed to satisfy a singular Lipschitz or a singular locally Lipschitz condition as follows:

- **Assumption** ($\mathcal{A}1$)**:** f is a singular Lipschitz function, i.e., there exist a positive number L_f and a non-negative number γ such that

$$|f(t, u(t)) - f(t, v(t))| \leq L_f t^{-\gamma}|u - v|,$$
$$|f(t, 0)| \leq L_f t^{-\gamma}$$

  for all $t \in (0, T]$, $u, v \in \mathbb{R}$.

- **Assumption** ($\mathcal{A}2$)**:** f is a singular locally Lipschitz function, i.e., for any $\Lambda > 0$, there exist a positive number $L_f(\Lambda)$ and a non-negative number γ such that

$$|f(t, u(t)) - f(t, v(t))| \leq L_f(\Lambda) t^{-\gamma}|u - v|,$$
$$|f(t, 0)| \leq L_f t^{-\gamma}$$

  for all $t > 0$ and $|u|, |v| \leq \Lambda$.

It is worth noting that differential equations where source functions satisfying Assumption ($\mathcal{A}1$) were considered in (Delbosco et al, 1996; Dien and Trong, 2021; Dien and Viet, 2021; Dien, 2021; Sin et al, 2016; Webb, 2019). However, in the mentioned papers, the authors only considered the cases $0 < \alpha \leq 1$ or $1 < \alpha \leq 2$. Besides, differential equations with source functions satisfying Assumption ($\mathcal{A}2$) are still not considered.

The topic of the continuous dependent of solutions of differential equations was studied in (Dien and Trong, 2021; Dien and Viet, 2021; Dien, 2021; Dien, 2022). This topic has been also studied for fractional partial differential equations such as (Dien and Trong, 2019; Trong et al, 2020; Trong el al, 2020; Viet et al, 2019). Usually, in these paper, the fractional orders of differential equations were restricted on the intervals $(0, 1]$ or $(1, 2]$.

Motivated by the above discussions, we consider the problem (1.1) with fractional order $\alpha > 0$ arbitrary and the source function satisfying the Assumptions ($\mathcal{A}1$) or ($\mathcal{A}2$). Our main contributions are that: (i) we prove that the problem has a unique maximal mild solution; (ii) we show that mild solutions of the problem are dependent continuously on fractional order and initial data; (iii) we prove that mild solutions of the problem blow-up at finite time under some appropriate conditions.

In section 2, we set up some notations, and introduce the concepts of fractional integral and Caputo fractional derivative. We also present preliminary lemmas, one of them may be new. In section 3, we divide it into two parts. The first part concerns the problem where the source function satisfies a singular Lipschitz condition. We include the poof of a known result on the existence and uniqueness of solutions of the problem. Besides, we give a simple proof of the continuous dependence of solutions of the problem on the fractional derivative. The second part is devoted to the problem with a locally singular Lipschitz source. We believe the results in this part are new.

2 Preliminaries

This section begin by setting up some notations. For $u \in C([0,T],\mathbb{R})$ and $0 < T_0 \leq T$, we define

$$\|u\| = \sup\{|u(t)| : 0 \leq t \leq T\} \quad \text{and} \quad \|u\|_{T_0} = \sup\{|u(t)| : 0 \leq t \leq T_0\}.$$

We also recall the classical Gamma and Beta functions

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,\mathrm{d}t, \quad B(p,q) = \int_0^1 (1-t)^{p-1}t^{q-1}\,\mathrm{d}t, \quad p,q > 0.$$

It is well-known that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.1}$$

Let us continue by presenting the concepts of the fractional integral and Caputo fractional derivative.

Definition 2.1. *The fractional integral of order $\alpha > 0$ is defined as*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)\,\mathrm{d}\tau$$

when the integral exists, and $I^0 h(t) = h(t)$.

Definition 2.2. *The Caputo derivative of fractional order $\alpha > 0$ is defined as*

$$D_t^\alpha h(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,\mathrm{d}s & \text{as } n-1 < \alpha < n = [\alpha]+1, \\ h^{(n)}(t) & \text{as } \alpha = n, \end{cases}$$

where $[\alpha]$ denotes the integer part of the real number α.

We now list here some properties of the fractional integral and fractional Caputo derivative.

Lemma 2.3 (Kilbas et al, 2006). *Let $n \in \mathbb{N}^*$ and $n-1 < \alpha \leq n$. We have*

(i). If $h \in C([0,T],\mathbb{R})$ then $D_t^\alpha(I^\alpha h(t)) = h(t)$.

(ii). If $h \in C^n([0,T],\mathbb{R})$ then $I^\alpha(D_t^\alpha h(t)) = f(t) + c_0 + c_1 t + .. + c_{n-1}t^{n-1}$.

(iii). $D_t^\alpha h(t) = 0$ implies that $h(t) = c_0 + c_1 t + .. + c_{n-1}t^{n-1}$.

(iv). For all $\alpha \geq \beta$, then $D_t^\beta I^\alpha h(t) = I^{\alpha-\beta}h(t)$.

To end this section, we give three lemmas that play an important role in the proof of the main results of the paper.

Lemma 2.4 (Dien, 2021). *Let $0 < p \leq 1$, and $q \leq p$, $q < 1$. For $0 \leq s \leq t \leq T$, we denote*

$$H_{p,q}(s,t) = \int_s^t (t-\tau)^{p-1}\tau^{-q}\,\mathrm{d}\tau, \quad H_{p,q}(t) = \int_0^t (t-\tau)^{p-1}\tau^{-q}\,\mathrm{d}\tau.$$

Then,

$$H_{p,q}(s,t) \to 0 \quad as \quad |s-t| \to 0, \;\; H_{p,q}(t) = t^{p-q}B\left(p, 1-q\right).$$

As a consequence,

$$H_{p,q}(t) \leq H_{p,q}(T)$$

for any $t \in [0,T]$.

Using the above Lemma, we can prove following result.

Lemma 2.5. *Assume that there exist $L_h > 0$ and $0 \leq \gamma \leq \alpha$, $\gamma < 1$ such that*

$$|h(t)| \leq L_h t^{-\gamma}$$

for all $t \in (0,T]$. Then, $I^\alpha h(t) \in C([0,T],\mathbb{R})$.

Proof. For any $\delta > 0$, by direct computation, we have

$$\begin{aligned}
&|I^\alpha h(t) - I^\alpha h(x+\delta)| \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t \left|(t-\tau)^{\alpha-1} - (t+\delta-\tau)^{\alpha-1}\right| |h(\tau)|\,\mathrm{d}\tau + \int_t^{t+\delta}(t+\delta-\tau)^{\alpha-1}|h(\tau)|\,\mathrm{d}\tau\right) \\
&\leq \frac{L_h}{\Gamma(\alpha)}\left(\int_0^t \left|(t-\tau)^{\alpha-1} - (t+\delta-\tau)^{\alpha-1}\right| t^{-\gamma}\,\mathrm{d}\tau + \int_t^{t+\delta}(t+\delta-\tau)^{\alpha-1}t^{-\gamma}\,\mathrm{d}\tau\right) \qquad (2.2)
\end{aligned}$$

To obtain the desired result, we consider two cases: $0 < \alpha \leq 1$ and $\alpha > 1$.

The first case: $0 < \alpha \leq 1$. It is obvious that $(t-\tau)^{\alpha-1} \geq (t+\delta-\tau)^{\alpha-1}$. So, we obtain from (2.2) together with Lemma 2.4 that

$$
\begin{aligned}
&|I^\alpha h(t) - I^\alpha h(x+\delta)| \\
&\leq \frac{L_h}{\Gamma(\alpha)} \left(\int_0^t \left((t-\tau)^{\alpha-1} - (t+\delta-\tau)^{\alpha-1} \right) t^{-\gamma} \, \mathrm{d}\tau + \int_t^{t+\delta} (t+\delta-\tau)^{\alpha-1} t^{-\gamma} \, \mathrm{d}\tau \right) \\
&= \frac{L_h}{\Gamma(\alpha)} \left(H_{\alpha,\gamma}(t) - H_{\alpha,\gamma}(t+\delta) + 2H_{\alpha,\gamma}(t, t+\delta) \right) \to 0 \text{ as } \delta \to 0.
\end{aligned}
$$

The second case: $\alpha > 1$. It is obvious that $(t-\tau)^{\alpha-1} \leq (t+\delta-\tau)^{\alpha-1}$. Using (2.2) and Lemma 2.4, we get

$$
\begin{aligned}
&|I^\alpha h(t) - I^\alpha h(x+\delta)| \\
&\leq \frac{L_h}{\Gamma(\alpha)} \left(\int_0^t \left((t+\delta-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1} \right) t^{-\gamma} \, \mathrm{d}\tau + \int_t^{t+\delta} (t+\delta-\tau)^{\alpha-1} t^{-\gamma} \, \mathrm{d}\tau \right) \\
&= \frac{L_h}{\Gamma(\alpha)} \left(H_{\alpha,\gamma}(t+\delta) - H_{\alpha,\gamma}(t) \right) \to 0 \text{ as } \delta \to 0.
\end{aligned}
$$

Thus, we conclude that $|I^\alpha h(t) - I^\alpha h(x+\delta)| \to 0$ as $\delta \to 0$ for all $\alpha > 0$ or $I^\alpha h(t) \in C([0,T], \mathbb{R})$. The proof of Lemma is finished. □

Lemma 2.6. *Let $\alpha > 0$ and $0 \leq \gamma < \min\{1, \alpha\}$. Suppose that u is a non-negative function and $u \in L^\infty[0,T]$, and u satisfies the following inequality*

$$
u(t) \leq C + \int_0^t C_1 (t-\tau)^{\alpha-1} \tau^{-\gamma} u(\tau)
$$

for some positive numbers A, B. Then, for any $0 < \rho < \min\{1-\gamma, \alpha-\gamma\}$, we have

$$
u(t) \leq 2^{1-\rho} C \exp\left(C_0 t \right),
$$

where $C_0 = 2^{1/\rho - 1} C_1^{1/\rho} B^{1/\rho-1} \left((\alpha-\rho)/(1-\rho), (1-\gamma-\rho)/(1-\rho) \right) T^{\alpha-\gamma-\rho}$.

Proof. The result of Lemma is a corollary of Theorem 1 in (Dien, 2022). □

3 Main results

In this section, we present the main results of the paper. First, using Lemma 2.3, we can easily transform the problem (1.1) to the following integral equation (also see (Diethelm, 2010))

$$\begin{aligned} u(t) &= \vartheta(t) + I^{\alpha} f(t, u(t)) \\ &= \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau))\, \mathrm{d}\tau, \end{aligned} \tag{3.1}$$

where

$$\vartheta(t) = \sum_{k=0}^{n-1} \xi_k \frac{t^k}{k!}.$$

Definition 3.1. *A function $u \in C([0,T], \mathbb{R})$ satisfying the integral equation (3.1) is called mild solution of the problem (1.1).*

Based on the obtained integral equation and the concept of mild solutions, we state and prove main results of the current paper. We divide the results in two cases: singular Lipschitz source, and singular locally Lipschitz source.

3.1 The case singular Lipschitz source

When the source function is a singular Lipschitz, we have the following results.

Theorem 3.2. *Let $\alpha > 0$ and $0 \le \gamma < \min\{1, \alpha\}$. Assume that* **Assumption** $(\mathcal{A}1)$ *holds. Then the problem (1.1) has a unique mild solution. Furthermore, for all ρ satisfying $0 < \rho < \min\{1-\gamma, \alpha-\gamma\}$, we have upper-bound estimate*

$$u(t) \le 2^{1-\rho} C \exp\left(C_0 t\right),$$

where

$$C = \sum_{k=0}^{n-1} |\xi_k| \frac{T^k}{k!} + \frac{L_f \Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)} T^{\alpha-\gamma}$$

and

$$C_0 = 2^{1/\rho - 1} \left(\frac{L_f}{\Gamma(\alpha)} \right)^{1/\rho} B^{1/\rho-1}\left((\alpha-\rho)/(1-\rho), (1-\gamma-\rho)/(1-\rho)\right) T^{\alpha-\gamma-\rho}.$$

Remark 3.3. *The existence and uniqueness of mild solutions of the fractional differential equations with the source functions satisfy* **Assumption** $(\mathcal{A}1)$ *were investigated in [**?**, **?**] for the cases $0 < \alpha \le 1$ and $1 < \alpha \le 2$. Herein, we prove the result for all $\alpha > 0$. Besides, we also present a new upper-bounded estimate for the solution of the problem.*

Proof. We consider the operator $\Phi : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ defined by

$$\Phi u(t) = \vartheta(t) + I^{\alpha} f(t, u(t)) = \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau))\, \mathrm{d}\tau, \tag{3.2}$$

where the function ϑ is defined in (3.1). From **Assumption** ($\mathcal{A}1$), we have $|f(t,u(t))| \leq L_f(|u|+1)t^{-\gamma} \leq M_u t^{-\gamma}$ where $M_u = L_f(\|u\|+1)$. Thus, in view of Lemma 2.5, we find that Φ is well-defined. We now show by induction that

$$|\Phi^m u(t) - \Phi^m v(t)| \leq \frac{C_m \Gamma(1-\gamma) L_f^m}{\Gamma(m(\alpha-\gamma)-\gamma+1)} t^{m(\alpha-\gamma)} \|u-v\|, \tag{3.3}$$

where

$$C_0 = 1,\ C_m = \frac{\Gamma(m(\alpha-\gamma)-\gamma+1)}{\Gamma(m(\alpha-\gamma)+1)} C_{m-1}.$$

Indeed, in view of Lemma 2.4 and (2.1), we have

$$\begin{aligned}
|\Phi u(t) - \Phi v(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau,u(\tau)) - f(\tau,u(\tau))|\, d\tau \\
&\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{-\gamma} |u(\tau)-u(\tau)|\, d\tau \\
&\leq \frac{L_f}{\Gamma(\alpha)} B(\alpha, 1-\gamma) t^{\alpha-\gamma} \|u-v\| \\
&= \frac{\Gamma(1-\gamma) L_f}{\Gamma(\alpha+1-\gamma)} t^{\alpha-\gamma} \|u-v\|.
\end{aligned}$$

Thus (3.3) holds for $m=1$. Suppose that (3.3) holds for $m=N$, then, we have

$$\begin{aligned}
&\left|\Phi^{N+1} u(t) - \Phi^{N+1} v(t)\right| \\
&\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{-\gamma} \left|\Phi^N u(\tau) - \Phi^N v(\tau)\right| d\tau \\
&\leq \frac{C_N \Gamma(1-\gamma) L_f^N}{\Gamma(\alpha)\Gamma(N(\alpha-\gamma)-\gamma+1)} L_f \|u-v\| \int_0^t (t-\tau)^{\alpha-1} \tau^{N(\alpha-\gamma)-\gamma}\, d\tau \\
&\leq \frac{C_N \Gamma(1-\gamma) L_f^N}{\Gamma(\alpha)\Gamma(N(\alpha-\gamma)-\gamma+1)} B(\alpha, N(\alpha-\gamma)+1-\gamma) t^{(N+1)(\alpha-\gamma)} \|u-v\| \\
&= \frac{C_{N+1} \Gamma(1-\gamma) L_f^{N+1}}{\Gamma((N+1)(\alpha-\gamma)-\gamma+1)} t^{(N+1)(\alpha-\gamma)} \|u-v\|.
\end{aligned}$$

It implies that (3.3) holds for all $m \in \mathbb{N}$. Since $\Gamma(x)$ is an increasing function for all $x \geq 2$, therefore, $\Gamma(m(\alpha-\gamma)-\gamma+1) \leq \Gamma(m(\alpha-\gamma)+1)$ for m large enough. This leads to $C_m = \frac{\Gamma(m(\alpha-\gamma)-\gamma+1)}{\Gamma(m(\alpha-\gamma)+1)} C_{m-1} \leq C_{m-1}$ for m large enough. So, we deduce that there exists $M > 0$ such that $0 < C_m < M$ for all $m \in \mathbb{N}$. Combining this fact with (3.3), we obtain

$$\|\Phi^m u - \Phi^m v\| \leq \frac{M\Gamma(1-\gamma) L_f^m}{\Gamma(m(\alpha-\gamma)-\gamma+1)} T^{m(\alpha-\gamma)} \|u-v\|.$$

It is easily to see that $\lim_{m\to\infty} \frac{M\Gamma(1-\gamma)L_f^m}{\Gamma(m(\alpha-\gamma)-\gamma+1)} T^{m(\alpha-\gamma)} = 0$, so, there exists an integer number m_0 such that $\frac{M\Gamma(1-\gamma)L_f^{m_0}}{\Gamma(m_0(\alpha-\gamma)-\gamma+1)} T^{m_0(\alpha-\gamma)} < 1/2$. This shows that Φ^{m_0} is a contraction mapping in $C([0,T],\mathbb{R})$. Consequently, Φ^{m_0} admits a unique fixed point $u \in C([0,T],\mathbb{R})$, i.e, $\Phi^{m_0} u = u$.

On the other hand, we have $\Phi u = \Phi^{m_0+1}u = \Phi^{m_0}(\Phi u)$. Thus, Φu is also the fixed point of Φ^{m_0}. From the uniqueness of fixed point of Φ^{m_0}, we conclude that $\Phi u = u$.

Let u be a unique mild solution of the problem (1.1), i.e., $\Phi u = u$. On the other hand, we have from **Assumption** ($\mathcal{A}1$) that $|f(t,u(t))| \le L_f(|u|+1)t^{-\gamma}$. This gives

$$\begin{aligned}|u(t)| &= |Qu(t)| \\ &\le \sum_{k=0}^{n-1} |\xi_k|\frac{T^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau))|\,\mathrm{d}\tau \\ &\le \sum_{k=0}^{n-1} |\xi_k|\frac{T^k}{k!} + \frac{L_f}{\Gamma(\alpha)}t^{\alpha-\gamma}B(\alpha,1-\gamma) + \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(|u(\tau)|+1\right)\,\mathrm{d}\tau \\ &\le C + \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\gamma-1}|u(\tau)|\,\mathrm{d}\tau,\end{aligned}$$

where $C = \sum_{k=0}^{n-1}|\xi_k|\frac{T^k}{k!} + \frac{L_f\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}T^{\alpha-\gamma}$. Here we have used the fact that $\frac{1}{\Gamma(\gamma)}B(\alpha,1-\gamma) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}$. Applying Lemma 2.6, we obtain the desired result. This completes the proof of Lemma. □

In the next theorem, we prove that the solution of the problem is dependent continuously on fractional order. To do that, let us consider approximate problem of the problem (1.1) as follows.

$$\begin{cases} D_t^{\alpha_j}u_j(t) = f(t,u_j(t)), & t\in(0,T] \\ u_j^{(k)}(0) = \xi_{k,j}, & (k=0,1,2,..,n-1).\end{cases} \tag{3.4}$$

Then, we have the following theorem.

Theorem 3.4. *Let $\alpha>0$, $\alpha_j>0$ for all $j\in\mathbb{N}^*$, and $0\le\gamma<\min\{1,\alpha,\inf_{j\in\mathbb{N}^*}\alpha_j\}$. Suppose that **Assumption** ($\mathcal{A}1$) is satisfied. Let u_α be a (unique) mild solution of the problem (1.1), and u_{α_j} be a (unique) mild solution of the approximate problem (3.4). If $\alpha_j\to\alpha$ and $\xi_{k,j}\to\xi_k$ as $k\to\infty$, then*

$$\left\|u_{\alpha_j}-u_\alpha\right\|\to 0 \quad as \quad j\to\infty.$$

Proof. Let us consider the operators $\Phi_j : C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ defined by

$$\Phi_j u(t) = \vartheta_j(t) + I^{\alpha_j}f(t,u(t)) = \vartheta_j(t) + \frac{1}{\Gamma(\alpha_j)}\int_0^t (t-\tau)^{\alpha_j-1}f(\tau,u(\tau))\,\mathrm{d}\tau, \tag{3.5}$$

where

$$\vartheta_j(t) = \sum_{k=0}^{n-1}\xi_{k,j}\frac{t^k}{k!}.$$

Using Lemma 2.6, we infer that Φ_j are well-defined for all $j\in\mathbb{N}$. Regarding to the operator Φ given by (3.2), by direct computation, we have

$$\left|\Phi_j u_{\alpha_j}(t) - \Phi u_\alpha(t)\right| \le \|\vartheta_j-\vartheta\| + \|P_j\| + \|Q_j\| + R_j(t), \tag{3.6}$$

where

$$
\begin{aligned}
P_j(t) &= \left|\frac{1}{\Gamma(\alpha_j)} - \frac{1}{\Gamma(\alpha)}\right| \int_0^t (t-\tau)^{\alpha-1} |f(\tau, u_\alpha(\tau))| \,\mathrm{d}\tau, \\
Q_j(t) &= \frac{1}{\Gamma(\alpha_j)} \int_0^t \left|(t-\tau)^{\alpha_j-1} - (t-\tau)^{\alpha-1}\right| |f(\tau, u_\alpha(\tau))| \,\mathrm{d}\tau, \\
R_j(t) &= \frac{1}{\Gamma(\alpha_j)} \int_0^t (t-\tau)^{\alpha_j-1} \big|f(\tau, u_{\alpha_j}(\tau)) - f(\tau, u_\alpha(\tau))\big| \,\mathrm{d}\tau.
\end{aligned}
$$

Step 1. Estimate for P_j. We have $|f(t, u_\alpha(t))| \le L_f(|u_\alpha| + 1)t^{-\gamma} \le M_{u_\alpha} t^{-\gamma}$ where $M_{u_\alpha} = L_f(\|u_\alpha\| + 1)$. In view of Lemma 2.4, we get

$$
\begin{aligned}
P_j(t) &\le M_{u_\alpha} \left|\frac{1}{\Gamma(\alpha_j)} - \frac{1}{\Gamma(\alpha)}\right| H_{\alpha,\gamma}(t) \\
&\le M_{u_\alpha} \left|\frac{1}{\Gamma(\alpha_j)} - \frac{1}{\Gamma(\alpha)}\right| H_{\alpha,\gamma}(T).
\end{aligned}
$$

Since $\Gamma(\alpha_j) \to \Gamma(\alpha)$ as $j \to \infty$, the latter inequality deduces that

$$
\|P_j\| \to 0 \quad \text{as} \quad j \to \infty.
$$

Step 2. Estimate for Q_j. Since $g(\tau) = (t-\tau)^{\alpha-1} - (t-\tau)^{\alpha_k-1}$ does not change sign on the interval $[0, t]$. Using the fact that $|f(t, u(t))| \le M_{u_\alpha} t^{-\gamma}$ and Lemma 2.4, we obtain

$$
\begin{aligned}
Q_j(t) &\le \frac{M_{u_\alpha}}{\Gamma(\alpha_j)} \left| \int_0^t \left((t-\tau)^{\alpha_j-1} - (t-\tau)^{\alpha-1}\right) t^{-\gamma} \,\mathrm{d}\tau \right| \\
&= \frac{M_{u_\alpha}}{\Gamma(\alpha_j)} \left| t^{\alpha_j-\gamma} B(\alpha_j, 1-\gamma) - t^{\alpha-\gamma} B(\alpha, 1-\gamma) \right| \to 0
\end{aligned}
$$

uniformly for all $t \in [0, T]$ as $j \to \infty$ or $\|Q_j\| \to 0$ as $j \to \infty$.

Step 3. Estimate for R_j.

$$
R_j(t) \le \frac{L_f}{\Gamma(\alpha_j)} \int_0^t (t-\tau)^{\alpha_j-1} t^{-\gamma} \big|u_{\alpha_j}(\tau) - u_\alpha(\tau)\big| \,\mathrm{d}\tau.
$$

Since $\Phi u_\alpha = u_\alpha$ and $\Phi_j u_{\alpha_j} = u_{\alpha_j}$, we can combine (3.6) and Step 3 to get

$$
\big|u_\alpha(t) - u_{\alpha_j}(t)\big| \le \|\vartheta_j - \vartheta\| + \|P_j\| + \|Q_j\| + \frac{L_f}{\Gamma(\alpha_j)} \int_0^t (t-\tau)^{\alpha_j-1} t^{-\gamma} \big|u_{\alpha_j}(\tau) - u_\alpha(\tau)\big| \,\mathrm{d}\tau.
$$

Fixed $0 < \rho < \inf_{j \in \mathbb{N}^*} \alpha_j$, applying Lemma 2.6, we obtain

$$
\begin{aligned}
\big|u_{\alpha_j}(t) - u_\alpha(t)\big| &\le 2^{1-\rho} \left(\|\vartheta_j - \vartheta\| + \|P_j\| + \|Q_j\|\right) \exp\left(C_j t\right) \\
&\le 2^{1-\rho} \left(\|\vartheta_j - \vartheta\| + \|P_j\| + \|Q_j\|\right) \exp\left(C_j T\right),
\end{aligned}
$$

where $C_j = 2^{1/\rho-1} \left(\frac{L_f}{\Gamma(\alpha_j)}\right)^{1/\rho} B^{1/\rho-1}\left((\alpha_j-\rho)/(1-\rho), (1-\gamma-\rho)/(1-\rho)\right) T^{\alpha_j-\gamma-\rho}$.

Using the fact that $\|\vartheta_j - \vartheta\| \to 0$, Step 1 and Step 2 together with the latter inequality, we conclude that $\left\|u_{\alpha_j}(t) - u_\alpha\right\| \to 0$ as $j \to \infty$. This finishes the proof of Theorem. □

3.2 The case singular locally Lipschitz source

Under the assumption the source is a singular locally Lipschitz, we obtain the existence, uniqueness and continuity results of maximal solution of the problem. We also get a result on the blow-up of solutions of the problem at the finite time. The results in this subsection seem to be new and have not been studied in previous papers.

Theorem 3.5. *Let $\alpha > 0$ and $0 \le \gamma < \min\{1, \alpha\}$, and let ϑ be defined in (3.1). Assume that* **Assumption** $(\mathcal{A}2)$ *is true. Then, for any $M > \|\vartheta\|$, we have*

(i). There exists $T_M > 0$ such that the problem (1.1) has a unique mild solution $u_\vartheta \in C([0, T_M], \mathbb{R})$.

(ii). If $u, \widetilde{u}$ are mild solutions of the problem (1.1) on $[0, T]$ then $u = \widetilde{u}$.

(iii). Put $T_\vartheta = \sup\{T :$ the problem (3.4) has a unique mild solution on$[0, T]\}$. The the problem (1.1) has a unique mild solution on $[0, T_M]$. Moreover, we have either $T_\vartheta = +\infty$ or $\lim_{t \to T_\vartheta^-} |u_\vartheta(t)| = +\infty$.

Proof. For $\Lambda > 0$, we define the function f_Λ as follows

$$f_\Lambda(t, u(t)) = f\left(t, \frac{\Lambda u(t)}{\max\{\Lambda, |u(t)|\}}\right).$$

We can easily to verify that

$$|f_\Lambda(t, u(t)) - f_\Lambda(t, v(t))| \le L_f(\Lambda) t^{-\gamma} |u - v| \tag{3.7}$$

for any $u, v \in \mathbb{R}$. We consider the following problem

$$v(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f_\Lambda(\tau, v(\tau))\, \mathrm{d}\tau. \tag{3.8}$$

From the singular Lipschitz condition (3.7) and Theorem 3.2, we find that the equation (3.8) has a unique solution $v_{\Lambda,T} \in C([0, T], \mathbb{R})$ for any $T > 0$.

(i). We choose $\Lambda = M$, then the equation (3.8) has a unique solution $v_{M,T} \in C([0, T], \mathbb{R})$ for any $T > 0$. Since $|v_{M,T}(0)| = |\vartheta(0)| < M$, by the continuity of the function $v_{M,T}$, we can find a number $T_M > 0$ such that $\|v_{M,T}\| \le M$ for any $t \in [0, T_M]$. This leads to $f_M(t, u(t)) = f(t, u(t))$ for all $t \in [0, T_M]$ and $u = v_{M,T}$ satisfies the equation (3.1), which is a mild solution of the problem (1.1). The proof of part (i) is done.

(ii). Assume that $u, \widetilde{u}$ are two mild solution of the problem (1.1) on $[0, T]$, we put

$$\Lambda = \max\{\|u\|, \|\widetilde{u}\|\}$$

and consider the equation (3.8). It is clear to see that $f_\Lambda(t,u(t)) = f(t,u(t))$ and $f_\Lambda(t,\widetilde{u}(t)) = f(t,\widetilde{u}(t))$ for all $t \in [0,T]$. These imply that $u, \widetilde{u}$ are solutions of the equation (3.8). By the uniqueness of solutions of the equation (3.8), we obtain $u(t) = \widetilde{u}(t)$ on $[0,T]$. This completes the proof of part (ii).

(iii). For $T \in (0, T_\vartheta)$, we denote by v_T the unique solution of the equation (3.8). For all $0 < T_1 < T_2 < T_\vartheta$, we find from part (ii) that $v_{T_1} = v_{T_2}$ on $[0,T_1]$. Thus, we can put $u_{T_\vartheta}(t) = v_T(t)$ for all $t \in [0,T]$ and $0 < T < T_\vartheta$. We obtain u_{T_ϑ} being unique mild solution of the problem (1.1) on $[0, T_\vartheta)$.

Next, we suppose that $T_\vartheta < +\infty$ and $\sup_{0\le t<T_\vartheta} |u_\vartheta(t)| < M < +\infty$. We consider the equation (3.8) with $\mu = M$ and denote its solution by u_{T^*} for some $T^* > T_\vartheta$. By the uniqueness of solution, we get $u_{T_\vartheta} = u_{T^*}$ for all $t \in [0, T_\vartheta)$. Since u_{T^*} is a continuous function, so, we can find $0 < \epsilon < T^* - T_\vartheta$ such that $\|u_{\epsilon+T_\vartheta}\| < M$. Hence, the problem (1.1) has a mild solution on $[0, \epsilon + T_\vartheta]$, which is a contraction with the definition of T_ϑ. So, if $T_\vartheta < +\infty$ then $\lim_{t\to T_\vartheta^-} |u_\vartheta(t)| = +\infty$. The proof of part (iii) is finished. □

Next, we show that under some appropriate assumptions mild solutions of the problem (1.1) blow-up at finite time. Indeed, we have the following theorem.

Theorem 3.6. *Let $0 < \alpha \le 1$ and $0 \le \gamma < \alpha$, and let ϑ be defined in (3.1). Assume that* **Assumption** $(\mathcal{A}2)$ *holds. Assume that $\inf_{t\ge 0} \vartheta(t) = \theta > 0$ and there exists $C > 0$ such that $f(t,u(t)) \ge C|u(t)|^p$ with $p > 1$. Then, the problem (1.1) admits a unique mild solution $u \in C([0,T_\vartheta],\mathbb{R})$. Furthermore, $|u(t)| \to +\infty$ as $t \to T_\vartheta^-$ and $T_\vartheta \le \left(\frac{\theta^{1-p}\Gamma(\alpha)}{C(p-1)}\right)^{1/\alpha}$.*

Proof. Using Theorem 3.5, we conclude that the problem (1.1) has a unique mild solution $u \in C([0,T_\vartheta],\mathbb{R})$. For any $0 < T < T_\vartheta$ and $0 < t \le T$, we have

$$\begin{aligned} u(t) &= \vartheta(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau))\, \mathrm{d}\tau \\ &\ge \theta + \frac{C}{\Gamma(\alpha)} T^{\alpha-1} \int_0^t |u(\tau)|^p \,\mathrm{d}\tau. \end{aligned}$$

Put

$$G(t) = \theta + \frac{C}{\Gamma(\alpha)} T^{\alpha-1} \int_0^t |u(\tau)|^p \,\mathrm{d}\tau.$$

We have $u(t) \ge G(t) \ge G(0) = \theta > 0$ and $G'(t) = \frac{C}{\Gamma(\alpha)} T^{\alpha-1} (u(t))^p \ge \frac{C}{\Gamma(\alpha)} T^{\alpha-1} (G(t))^p$. The latter inequality implies that

$$\frac{1}{p-1}\left((G(0))^{1-p} - (G(t))^{1-p} \right) \ge \frac{C}{\Gamma(\alpha)} T^{\alpha-1} t.$$

Letting $t = T$, we obtain from the latter inequality that

$$\frac{1}{p-1}(G(0))^{1-p} \geq \frac{C}{\Gamma(\alpha)}T^{\alpha}.$$

This leads to

$$T \leq \left(\frac{\theta^{1-p}\Gamma(\alpha)}{C(p-1)}\right)^{1/\alpha}.$$

Letting $T \to T_\vartheta$, we obtain

$$T_\vartheta \leq \left(\frac{\theta^{1-p}\Gamma(\alpha)}{C(p-1)}\right)^{1/\alpha}.$$

This completes the proof of Theorem. □

Theorem 3.7. *Let $\alpha > 0$ and $0 \leq \gamma < \min\{1, \alpha\}$, and let ϑ and ϑ_j be defined in (3.1) and (3.5), respectively. Assume that* **Assumption** $(\mathcal{A}2)$ *holds. Assume further that T_ϑ, T_{ϑ_j} are the maximal existence interval of the equation (3.1) and (3.5), respectively. Then, for any $T_0 \in (0, T_\vartheta)$, there exists $j_0 > 0$ such that $T_{\vartheta_j} > T$ for any $j \geq j_0$. Moreover*

$$u_{\alpha_j} \to u_\alpha \quad in \quad C([0, T_0], \mathbb{R}).$$

Proof. For $\Lambda > \|u_\alpha\|$, we define

$$f_\Lambda(t, u(t)) = f\left(t, \frac{\Lambda u(t)}{\max\{\Lambda, |u(t)|\}}\right).$$

We have $f_\Lambda(t, u(t)) = f(t, u(t))$ for all $\|u\| \leq \Lambda$, and

$$|f_\Lambda(t, u(t)) - f_\Lambda(t, v(t))| \leq L_f(\Lambda)t^{-\gamma}|u - v|.$$

We consider two following equations

$$u_{\Lambda,\alpha}(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f_\Lambda(\tau, u_{\Lambda,\alpha}(\tau))\,\mathrm{d}\tau, \tag{3.9}$$

$$u_{\Lambda,\alpha_j}(t) = \vartheta(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f_\Lambda(\tau, u_{\Lambda,\alpha_j}(\tau))\,\mathrm{d}\tau. \tag{3.10}$$

We infer from theorem 3.2 that the equations (3.9) and (3.10) have unique solution in $C([0, \infty), \mathbb{R})$. In view of Theorem 3.4, we have

$$u_{\Lambda,\alpha_j} \to u_{\Lambda,\alpha} \quad \text{in} \quad C([0, T], \mathbb{R}). \tag{3.11}$$

Since $\|u_\alpha\| < \Lambda$, one gets $f_\Lambda(\tau, u_\alpha(\tau)) = f(\tau, u_\alpha(\tau))$. By the uniqueness result, we obtain $u_{\Lambda,\alpha}(t) = u_\alpha(t)$ on $[0, T_0]$. From (3.11) and $\|u_\alpha\| < \Lambda$, we can choose j_0 such that $\left\|u_{\Lambda,\alpha_j}\right\| < \Lambda$ for any $j \geq j_0$. So, $f_\Lambda(\tau, u_{\Lambda,\alpha_j}(\tau)) = f(\tau, u_{\Lambda,\alpha_j}(\tau))$ and $u_{\Lambda,\alpha_j} = u_{\alpha_j}$ is solution of equation (3.5) as $j \geq j_0$ and $\left\|u_{\Lambda,\alpha_j}\right\| < \Lambda$. Thus, one has $T_0 < T_{\vartheta_j}$ for $j \geq j_0$. Using (3.11, we obtain the desired result of Theorem. □

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