ĐẠI HỌC THỦ DẦU MỘT
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On existence results for a nonlinear differential equation involving Caputo-Katugampola fractional derivative with a nonlocal initial condition

by ***Bui Thi Ngoc Han***, ***Nguyen Thi Linh*** *(Thu Dau Mot university)*

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Corresponding author: ntinh@tdmu.edu.vn (Nguyen Thi Linh)
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ABSTRACT

This paper is devoted to study a fractional equation involving Caputo-Katugampola derivative with nonlocal initial condition. Unlike previous papers, in this paper, the source function of problem is assumed having a singularity. We propose some reasonable conditions such that the problem has at least one mild solution or has a unique mild solution. The desired results are proved by using the Banach, Leray-Schauder and Krasnoselskii fixed point theorems. Some examples are given to confirm our theoretical findings.

Keywords: *Caputo-Katugampola fractional derivative; Nonlinear integral equations; existence MSC[2010] 26A33; 35A01; 35A02; 35R11.*

1 Introduction

The subject of fractional calculus has applications in diverse and widespread fields of science and engineering such as physics, quantum mechanics, bioengineering, etc, we refer to (Podlubny, 1999; Samko, Kilbas and Marichev, 1987; Diethelm, 2010; Herrmann, 2014; Iomin, 2019; Magin, 2006; Tarasov, 2010; Uchaikin, 2013) and the references therein.

Study the existence is one of the important topics in fractional differential equations. There are various papers that investigate on existence results for the fractional differential equations with Caputo, Caputo-Hadamard, and Caputo-Katugampola derivative (Redhwan et al., 2019; Hamad and Ntouyas, 2017; Benchohra et al., 2008; Gu et al., 2019; Da C. Sousa et al., 2016). However, in the mentioned papers the authors have used the globally Lipschitz conditions, i.e.,

$$|f(t,x) - f(t,y)| \le k(t)|x-y|$$

or

$$|f(t,x)| \le k(t),$$

where k is a continuous function in $[0, T]$. Problems with source functions satisfy the following non-globally Lipschitz conditions

$$|f(t,x) - f(t,y)| \leq \kappa t^{-p}|x-y| \text{ or } |f(t,x)| \leq \kappa t^{-q}$$

is still not study. Besides, we can not find any paper deal with existence results for the problem involving Caputo-Katugampola derivative with nonlocal initial condition.

Motivated by these reasons, the current paper consider the following problem with Caputo-Katugampola derivative

$$^{C}D^{\alpha,\rho}_{0+}x(t) = f(t, x(t)),\ t \in (0, T],\ \alpha \in (0, 1),\ \rho > 0 \tag{1.1}$$

subject to the nonlocal initial condition

$$x(0) = \int_0^T g(\tau, x(\tau))\,\mathrm{d}\tau, \tag{1.2}$$

where $f, g \in C((0,T) \times \mathbb{R}, \mathbb{R})$.

In the next section, according to the Banach, Leray-Schauder and Krasnoselskii fixed point theorems, we introduce three existence results for our problem.

2 Preliminaries

In this section, we introduce some notations, definitions and some essential lemmas which we will use in the proof of main results of our paper.

We firstly set up some notations that use throughout the rest of the paper. For $x \in C([0,T], \mathbb{R})$, we denote the sup-norm by $\|x\| := \sup_{0\leq t\leq T} |x(t)|$. We also remind the Gamma and Beta functions

$$\Gamma(p) = \int_0^\infty s^{p-1}e^{-s}\,\mathrm{d}s, \quad B(p,q) = \int_0^1 (1-s)^{p-1}s^{q-1}\,\mathrm{d}s,\ (p,q > 0).$$

Note that, we have the following identity

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.1}$$

Secondly, we present definitions of the integral Katugampola and Caputo-Katugampola fractional derivative. These definitions readers can find in (R. Almeida, A. B. Malinowska and T. Odzijewicz, 2016; R. Almeida, 2017) and the references therein. We start with defining the Katugampola fractional integrals as follows.

Definition 2.1. *Let $\alpha \in (0,1)$, $\rho > 0$, $0 \leq a < b < +\infty$, and let x be an integrable function on $[a, b]$. The Katugampola fractional integrals is defined by*

$$I^{\alpha,\rho}_{a+}x(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} x(\tau)\,\mathrm{d}\tau.$$

Now we define the Caputo-Katugampola fractional derivative.

Definition 2.2. *Let $\alpha \in (0,1)$, $\rho > 0$, $0 \le a < b < +\infty$, and let x be an integrable function on $[a,b]$. The Caputo-Katugampola fractional derivative is defined by*

$$ {}^C D_{a+}^{\alpha,\rho} x(t) = \frac{d}{dt} I_{a+}^{1-\alpha,\rho} = \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} x(\tau)\, \mathrm{d}\tau. $$

To end this section, we state and prove some essential lemmas which we will use in proof of main results of our paper.

Lemma 2.3. *Let $\alpha \in (0,1)$, $\rho > 0$. If $\gamma < 1$ then*

$$ \int_0^t \frac{\tau^{-\gamma}}{(t^\rho - \tau^\rho)^{1-\alpha}}\, \mathrm{d}\tau = \frac{1}{\rho} B\left(\alpha, \frac{1-\gamma}{\rho}\right) t^{\rho(\alpha-1)+1-\gamma} \text{ for any } t \in [0,T] $$

and

$$ \int_{t_1}^{t_2} \frac{\tau^{-\gamma}}{(t_2^\rho - \tau^\rho)^{1-\alpha}}\, \mathrm{d}\tau \le C t_2^{\rho(\alpha-1)+1-\gamma} \max\left\{ \left(1 - \left(\frac{t_1}{t_2}\right)^\rho\right)^\alpha, 1 - \left(\frac{t_1}{t_2}\right)^{\frac{1-\gamma}{\rho}} \right\}, $$

where $C = \frac{1}{\rho} \max\left\{1, (1/2)^{\frac{1-\gamma}{\rho}-1}, (1/2)^{\alpha-1}\right\}$. Consequently, if $\rho(\alpha-1)+1-\gamma > 0$ then

$$ \int_{t_1}^{t_2} \frac{\tau^{-\gamma}}{(t_2^\rho - \tau^\rho)^{1-\alpha}}\, \mathrm{d}\tau \to 0 \text{ uniformly as } t_1 \to t_2 \text{ in } [0,T]. $$

Proof. By putting $s = (\tau/t_2)^\rho$ and direct computation, we can easy to verify that

$$ \int_{t_l}^{t_2} \frac{\tau^{-\gamma}}{(t_2^\rho - \tau^\rho)^{1-\alpha}}\, \mathrm{d}\tau = \frac{1}{\rho} t_2^{\rho(\alpha-1)+1-\gamma} \int_{(t_1/t_2)^\rho}^1 (1-s)^{\alpha-1} s^{\frac{1-\gamma}{\rho}-1}\, \mathrm{d}s \text{ for any } t_1 < t_2. \tag{2.2} $$

This leads to the first result of Lemma. To get the second result, we divide into two cases, the first case is $(t_1/t_2)^\rho \ge 1/2$ and the second case is $(t_1/t_2)^\rho < 1/2$. By using (2.2), we obtain the desired result. □

Lemma 2.4. *The problem (1.1) and (1.2) is equivalent to the integral equation*

$$ x(t) = \int_0^T g(\tau, x(\tau))\, \mathrm{d}\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau))\, \mathrm{d}\tau. \tag{2.3} $$

Proof. Almeida et al (D.R.,1980) shown that the solution of the equation (1.1) with the initial condition $x(0) = x_0$ is equivalent to the Volterra integral equation

$$ x(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau))\, \mathrm{d}\tau. $$

By the nonlocal initial condition (1.2), we conclude that the problem (1.1) and (1.2) are equivalent to the integral equation (2.3). □

Remark 2.5. *We note that the equation (2.3) is nonlocal, i.e, the integral is defined in all interval $[0,T]$. Therefore, we can not apply the technique that used in [**?**] to study the existence solutions of our problem.*

Theorem 2.6. *Let $\alpha \in (0,1)$, $\rho > 0$. Let $f, g \in C((0,T)\times\mathbb{R},\mathbb{R})$. Suppose that there exist two positive constants K_1, K_2, and two numbers p, q with $p < \alpha\rho$ and $q < 1$ such that*

$$|f(t,x)-f(t,y)| \le K_1 t^{-p}|x-y| \text{ and } |g(t,x)-g(t,y)| \le K_2 t^{-q}|x-y|$$

for all $u, v \in C([0,T],\mathbb{R})$. If $|f(t,0)| \le Kt^{-r}$ for some constants $K > 0$, $r < \alpha\rho$ and

$$K_1 \frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)} + K_2\frac{T^{1-q}}{1-q} < 1$$

then the problem (1.1)-(1.2) has a unique solution in $C([0,T],\mathbb{R})$.

Proof. For $x \in C([0,T],\mathbb{R})$, let us put

$$Fx(t) = \int_0^T g(\tau,x(\tau))\,\mathrm{d}\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}} f(\tau,x(\tau))\,\mathrm{d}\tau. \tag{2.4}$$

For $t_2 > t_1$ and for each $x \in C([0,T],\mathbb{R})$ with $\|x\| \le M$, we have $(t_1^\rho-\tau^\rho)^{1-\alpha} \le (t_2^\rho-\tau^\rho)^{1-\alpha}$ and $|f(t,x)| \le K_1t^{-p}|x| + |f(t,0)| \le K_1Mt^{-p}+Kt^{-r}$. This deduces

$$\begin{aligned}
|Fx(t_1)-Fx(t_2)| \le\ & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}\left(\frac{\tau^{\rho-1}}{(t_1^\rho-\tau^\rho)^{1-\alpha}} - \frac{\tau^{\rho-1}}{(t_2^\rho-\tau^\rho)^{1-\alpha}}\right)|f(\tau,x(\tau))|\,\mathrm{d}\tau \\
+\ & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\tau^{\rho-1}}{(t_2^\rho-\tau^\rho)^{1-\alpha}}|f(\tau,x(\tau))|\,\mathrm{d}\tau \\
\le\ & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}\frac{\tau^{\rho-1}}{(t_1^\rho-\tau^\rho)^{1-\alpha}}\left(K_1M\tau^{-p}+K\tau^{-r}\right)\mathrm{d}\tau \\
-\ & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}\frac{\tau^{\rho-1}}{(t_2^\rho-\tau^\rho)^{1-\alpha}}\left(K_1M\tau^{-p}+K\tau^{-r}\right)\mathrm{d}\tau \\
+\ & \frac{2\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\tau^{\rho-1}}{(t_2^\rho-\tau^\rho)^{1-\alpha}}\left(K_1M\tau^{-p}+K\tau^{-r}\right)\mathrm{d}\tau.
\end{aligned}$$

Using Lemma 2.3 with $\gamma = p-\rho+1$ and $\gamma = r-\rho+1$, we obtain

$$\begin{aligned}
|Fx(t_1)-Fx(t_2)| \le\ & \frac{K_1M\rho^{-\alpha}}{\Gamma(\alpha)}B\left(\alpha,\frac{\rho-p}{\rho}\right)\left(t_2^{\alpha\rho-p}-t_1^{\alpha\rho-p}\right) \\
+\ & \frac{K\rho^{-\alpha}}{\Gamma(\alpha)}B\left(\alpha,\frac{\rho-r}{\rho}\right)\left(t_2^{\alpha\rho-r}-t_1^{\alpha\rho-r}\right) \\
+\ & \frac{2C_1K_1M\rho^{-\alpha}}{\Gamma(\alpha)}t_2^{\rho\alpha-p}\max\left\{\left(1-\left(\frac{t_1}{t_2}\right)^\rho\right)^\alpha, 1-\left(\frac{t_1}{t_2}\right)^{\frac{\rho-p}{\rho}}\right\} \\
+\ & \frac{2C_2K\rho^{-\alpha}}{\Gamma(\alpha)}t_2^{\rho\alpha-r}\max\left\{\left(1-\left(\frac{t_1}{t_2}\right)^\rho\right)^\alpha, 1-\left(\frac{t_1}{t_2}\right)^{\frac{\rho-r}{\rho}}\right\},
\end{aligned}$$

where C_1, C_2 independent of t_1 and t_2. Since $r, p < \alpha\rho < \rho$, the last inequality lead to

$$|Fx(t_1)-Fx(t_2)| \to 0 \text{ (uniformly) as } t_1 \to t_2 \text{ on } [0,T]. \tag{2.5}$$

This shows that F is the mapping from $C([0,T],\mathbb{R})$ into itself.

Now, by direct computation, we have

$$\begin{aligned} &|Fx(t)-Fy(t)| \\ \leq\ & K_2\int_0^T t^{-q}|x(\tau)-y(\tau)|\,\mathrm{d}\tau + K_1\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-p-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}|x(\tau)-y(\tau)|\,\mathrm{d}\tau \\ \leq\ & \|x-y\|\left(K_2\int_0^T t^{-q}\,\mathrm{d}\tau + K_1\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{-(p-\rho+1)}}{(t^\rho-\tau^\rho)^{1-\alpha}}\,\mathrm{d}\tau\right). \end{aligned}$$

We can use Lemma 2.3 with $\gamma = p+1-\rho$ and the identity (2.1) to get that

$$\begin{aligned} |Fx(t)-Fy(t)| &\leq \left(K_1\frac{\rho^{-\alpha}}{\Gamma(\alpha)}B\left(\alpha,\frac{\rho-p}{\rho}\right)t^{\alpha\rho-p} + K_2\frac{T^{1-q}}{1-q}\right)\|x-y\| \\ &= \left(K_1\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p} + K_2\frac{T^{1-q}}{1-q}\right)\|x-y\| \end{aligned}$$

This implies that F is contraction mapping in $C([0,T],\mathbb{R})$. Consequently, the problem (1.1) and (1.2) has a unique solution in $C([0,T],\mathbb{R})$. □

Theorem 2.7. *Let $\alpha\in(0,1)$, $\rho>0$, and $p<\alpha\rho$, $q<1$. Let $f,g\in C((0,T)\times\mathbb{R},\mathbb{R})$. Suppose that there exist two positive and increasing functions $\varphi,\psi:[0,+\infty)\to[0,+\infty)$, and two positive constants $K_1,K_2>0$ such that*

$$|f(t,x)|\leq K_1t^{-p}\varphi(|x|),\quad |g(t,x)|\leq K_2t^{-q}\psi(|x|).$$

If there exists a positive constant Λ such that

$$\Lambda > K_1\varphi(\Lambda)\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p} + K_2\psi(\Lambda)\frac{T^{1-q}}{1-q}. \tag{2.6}$$

Then, the problem (1.1)-(1.2) has at least one solution in $C([0,T],\mathbb{R})$.

Remark 2.8. *If*

$$\varphi(s)=\sum_{i=1}^n a_is^{p_i},\quad \psi(s)=\sum_{j=1}^m b_js^{q_j},\quad (a_i,b_j\in\mathbb{R},\quad p_i,q_j\in[0,1))$$

then the assumption (2.6) holds.

Proof. Let us consider the operator F which defined in (2.4). Put

$$W=\{z\in C([0,T],\mathbb{R}):\|z\|\leq\Lambda\}.$$

We will show that F is completely continuous. In fact, we put $M=\frac{T}{1-q}+\frac{T^{\rho\alpha-p}}{\rho^\alpha\Gamma(\alpha)}B(\alpha,\frac{\rho-p}{\rho})$. For any $\epsilon>0$, there exists $\delta>0$ such that $t^p|f(t,x(t))-f(t,y(t))|<\epsilon/M$ and $t^q|g(t,x(t))-g(t,y(t))|<$

ϵ/M for any $\|x-y\|<\delta$. Applying Lemma 2.3 with $\gamma:=\rho-1$, one has

$$\begin{aligned}|Fx(t)-Fy(t)| &\le \int_0^T |g(\tau,x(\tau))-g(\tau,y(\tau))|\,\mathrm{d}\tau \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}|f(\tau,x(\tau))-f(\tau,y(\tau))|\,\mathrm{d}\tau \\ &< \frac{\epsilon}{M}\left(\int_0^T t^{-q}\,\mathrm{d}\tau+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-p-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\,\mathrm{d}\tau\right) \\ &= \frac{\epsilon}{M}\left(\frac{T}{1-q}+\frac{T^{\rho\alpha-p}}{\rho^\alpha\Gamma(\alpha)}B(\alpha,\frac{\rho-p}{\rho})\right)=\epsilon\end{aligned}$$

due to $B(\alpha,1)=1/\alpha$. This implies $\|Fx-Fy\|<\epsilon$ or F is continuous.

For $x\in C([0,T],\mathbb{R})$ with $\|x\|\le E$, applying Lemma 2.3 with $\gamma=p-\rho+1$, and direct computation, we have

$$\begin{aligned}|Fx(t)| &\le K_1\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \varphi(|x(\tau)|)\frac{\tau^{-(p-\rho+1)}}{(t^\rho-\tau^\rho)^{1-\alpha}}\,\mathrm{d}\tau+K_2\int_0^T\psi(|x(\tau)|)\tau^{-q}\,\mathrm{d}\tau \\ &\le K_1\varphi(E)\frac{\rho^{-\alpha}}{\Gamma(\alpha)}B\left(\alpha,\frac{\rho-p}{\rho}\right)T^{\alpha\rho-p}+K_2\psi(E)\frac{T^{1-q}}{1-q} \\ &= K_1\varphi(E)\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p}+K_2\psi(E)\frac{T^{1-q}}{1-q}. \end{aligned}\qquad (2.7)$$

This shows that F is bounded. Lastly, similar to the proof of (2.5), we can prove that F is equicontinuous. Consequently, F is compact operator.

We suppose that there exists $x\in\partial W$ and $\lambda\in(0,1)$ such that $x=\lambda Fx$. Similarly the proof of (2.7), we have

$$\Lambda=\|x\|=\lambda\|Fx\|\le K_1\varphi(\Lambda)\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p}+K_2\psi(\Lambda)\frac{T^{1-q}}{1-q}.$$

The last inequality is contradiction with (2.6). Applying the nonlinear Leray-Schauder alternatives fixed point theorem (A. Granas, 2003), we obtain the result of Theorem. □

Theorem 2.9. *Let $\alpha\in(0,1)$, $\rho>0$, $p<\alpha\rho$. Let $f,g\in C((0,T)\times\mathbb{R},\mathbb{R})$. Suppose that there exist three constants $p<\alpha\rho$, and $q,r<1$ such that*

$$|g(t,x)-g(t,y)|\le Kt^{-r}|x-y| \qquad (2.8)$$

and

$$|f(t,x)|\le Pt^{-p},\quad |g(t,x)|\le Qt^{-q}$$

for some positive numbers K, P, Q and for any $x,y\in C([0,T],\mathbb{R})$. If

$$KT^{1-r}/(1-r)<1 \qquad (2.9)$$

then the problem (1.1)-(1.2) has at least one solution in $C([0,T],\mathbb{R})$.

Remark 2.10. *The result of Theorem 2.9 holds if we replace the assumption (2.8) by*

$$|f(t,x)-f(t,y)| \leq Kt^{-r}|x-y| \text{ and } |f(t,0)| \leq Kt^{-r}, \ (r<\alpha\rho)$$

and (2.9) by $K\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-r}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-r}{\rho}\right)}T^{\alpha\rho-r}<1$.

Proof. For $x,y \in C([0,T],\mathbb{R})$, we define

$$Ax(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}f(\tau,x(\tau))\,\mathrm{d}\tau, \quad \mathcal{B}y(t)=\int_0^T g(\tau,y(\tau))\,\mathrm{d}\tau.$$

Let us put

$$\mathcal{D}=\left\{z\in C([0,T],\mathbb{R}) : \|z\| \leq \theta := P\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p}+Q\frac{T^{1-q}}{1-q}\right\}.$$

By the same method that used in Theorem 2.6, we can verify that

$$|Ax(t_1)-Ax(t_2)| \to 0 \ \text{ as } \ t_1\to t_2.$$

Hence A is the mapping from $(C[0,T],\mathbb{R})$ into itself. Also, by the same manner in Theorem 2.7, we can prove that A is compact operator. Moreover, one has the following estimation

$$\begin{aligned}|Ax(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}|f(\tau,x(\tau))|\,\mathrm{d}\tau \\ &\leq P\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-p-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\,\mathrm{d}\tau.\end{aligned}$$

Applying Lemma 2.3 with $\gamma=p-\rho+1$, we obtain

$$|Ax(t)| \leq P\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p}.$$

This implies

$$\|Ax\| \leq P\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha+\frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p}. \tag{2.10}$$

Obviously, $\mathcal{B}$ is a mapping from $C([0,T],\mathbb{R})$ into itself. We will verify that $\mathcal{B}$ is contraction. Indeed, according to assumption (2.8), we have

$$\begin{aligned}|\mathcal{B}x(t)-\mathcal{B}y(t)| &\leq \int_0^T |g(\tau,x(\tau))-g(\tau,y(\tau))|\,\mathrm{d}\tau \\ &\leq K\int_0^T \tau^{-r}|x(\tau)-y(\tau)|\,\mathrm{d}\tau \\ &\leq \frac{KT^{1-r}}{1-r}\|x-y\|.\end{aligned}$$

Since $\frac{KT^{1-r}}{1-r} < 1$, the last inequality implies that A is contraction. On the other hand, we have the estimation

$$|\mathcal{B}x(t)| \leq \int_0^T |g(\tau, x(\tau))| \leq Q \int_0^T \tau^{-q}\, \mathrm{d}\tau = \frac{QT^{1-q}}{1-q}.$$

This implies

$$\|\mathcal{B}x\| \leq \frac{QT^{1-q}}{1-q}. \tag{2.11}$$

Combining the inequality (2.10) with (2.11), we obtain

$$\|Ax + \mathcal{B}y\| \leq \|Ax\| + \|\mathcal{B}y\| \leq P\frac{\rho^{-\alpha}\Gamma\left(\frac{\rho-p}{\rho}\right)}{\Gamma\left(\alpha + \frac{\rho-p}{\rho}\right)}T^{\alpha\rho-p} + \frac{QT^{1-q}}{1-q} = \theta.$$

This shows that $Ax + \mathcal{B}y \in \mathcal{D}$ for any $x, y \in \mathcal{D}$. Applying the Krasnoselskii fixed point theorem (D.R., 1980), we obtain the desired result of Theorem. □

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