A UNIQUENESS CRITERIA FOR ORDINARY FIRST ORDER DIFFERENTIAL EQUATIONS

Ngo Le Hong Phuc(1)
(1) Thu Dau Mot university
Corresponding author: phucnlh@tdmu.edu.vn

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Abstract

The theory of differential equations arises from the study of physical phenomena. This field has various applications in science and engineering. The study of qualitative properties for each mathematical model plays an important role, attracting the attention of both theoretical and applied researchers. Normally, the most significant qualitative property to be studied first is the existence and uniqueness of the solutions of each mathematical model. However, proving existence and uniqueness results for mathematical models where the source function has a singularity is a difficult problem and requires many different techniques. In this paper, we establish some new conditions suitable to achieve the unique solution criterion for ordinary first-order differential equations. To obtain the desired results, we have improved the methods that have been used to prove the results in the work of Krasnosel'skii and Krein (Krasnoselskii and Krein, 1956). In addition, we also provide an example to illustrate the theoretical results.

Keywords: differential equations, lipschitz condition, uniqueness

1. Introduction

We consider the initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

where the function $f(x, y)$ is at least continuous in a domain $D \subseteq \mathbb{R}^2$, and $(x_0, y_0) \in D$. By a solution of (1) in an interval J containing x_0, we mean a function $y(x)$ satisfying:

(i) $y(x_0) = y_0$,

(ii) for all $x \in J$, the points $(x, y(x)) \in D$

(iii) $y'(x)$ exists and continuous for all $x \in J$

(iv) $y'(x) = f(x, y(x))$.

If J is closed then at the endpoints of J only the one-sided existence of $y'(x)$ is assumed. It is well known that the continuity of $f(x,y)$ in a closed rectangle

$$\bar{S}: |x - x_0| \le a, |y - y_0| \le b$$

is sufficient for the existence of at least one solution of (1) in the interval $J_h: |x - x_0| \le h = min(a, b/M)$, where $M = sup_{\bar{S}}|f(x,y)|$. Although the continuity of $f(x,y)$ is sufficient for the existence of a solution of (1), it does not imply the uniqueness of the solutions.

We know that the significance of uniqueness theorems in the study of initial value problems is well-known due to their relevance in establishing the well-posedness of the real-world problems arising in physical, and engineering systems. Uniqueness results play a significant role in the continuation of solutions and the theory of autonomous systems. Meanwhile, the uniqueness results almost always come with the cost of stringent conditions, they are valuable, for without such uniqueness results it is impossible to make predictions about the behavior of physical systems. Hence, there are many studies on sufficient conditions for the unique solution of differential equations. In particular, in cases where the source function does not satisfy the usual Lipschitz condition and contains a singularity, the conditions for the problem to have a unique solution often become more difficult.

2. Literature Review

The pioneering work of Nagumo (Nagumo, 1926) opened a new trend of studying the theory of differential equations in which source functions have singularities. In the mentioned work, the author showed that problem (1) where the source function satisfies the following condition

$$|f(x,y) - f(x,\bar{y})| \le K|y - \bar{y}|^{-1} \quad (2)$$

has at most one solution. Continuously the work of Nagumo, there are many uniqueness criteria have been proposed (e.g., Athanassov, 1990; Biles and Spraker, 2014; Ferreira, 2013; Gard, 1978; Constantin, 2010; Markus, 1953; Mejstrik, 2012; Yifei and Mei, 2010). Krasnoselskii and Krein (1956), the authors considered the problem (1) with the source function satisfying the condition (2) and proved that if

$$|f(x,y) - f(x,\bar{y})| \le K|y - \bar{y}|^{\alpha} \quad (3)$$

then the problem has a unique solution. It is worth noting that condition (3) implies that the source function may not have a singularity. To the best of our knowledge, the source function satisfies the weaker condition as follows

$$|f(x,y) - f(x,\bar{y})| \le \frac{m}{(x - x_0)^{\gamma}}|y - \bar{y}|^{\alpha}, \qquad x > x_0$$

is still not considered.

3. Methods

Inspired by the previous works in the literature, in this paper, we improve some conditions in the work of Krasnosel'skii and Krein (1956) for the problem (1). More precisely, we show that the condition (3) can be replaced by the weaker condition (4). It is worth mentioning that this condition means f may have a singularity at $x = x_0$.

In this paper, we improve the method used in the work of Krasnosel'skii and Krein (1956). First, we transform the problem into an integral equation, and then we use the contradiction principle to show that the problem has at most one solution.

The remainder of the present paper is outlined as follows. In section 2, we state and prove the main result of the paper. In section 3, a befitting example is structured to show the applicability of the theoretical result.

4. Results

In this section, we establish an additional condition to achieve the uniqueness of the solutions of the initial value problem (1).

4.1. Uniqueness result

Theorem 2.1. Let $f(x,y)$ be continuous in $\bar{S}$ and for all $(x,y),(x,\bar{y}) \in \bar{S}$ it satisfies

$$|f(x,y) - f(x,\bar{y})| \le \boldsymbol{k}|x - x_0|^{-\mathbf{1}}|y - \bar{y}|, x \neq x_0, k > 1 \tag{3}$$

and the condition

$$|f(x,y) - f(x,\bar{y})| \le \frac{m}{(x-x_0)^\gamma}|y - \bar{y}|^\alpha, m > 0,\ \gamma < 1, 0 < \alpha < 1, \gamma + k(1-\alpha) < 1 \tag{4}$$

(k(1$-\alpha$) $<$ 1 is no restriction at all,when k$\le$ 1.)

Then, the initial value problem (1) has at most one solution in $|x - x_0| \le a$.

To prove the main result, we use the following lemma.

Lemma 2.1 (see [4]). Let $f(x,y)$ be continuous in the domain D, then any solution of (1) is also a solution of the integral equation

$$y(x) = y_{\mathbf{0}} + \int_{x_0}^{x} f(t, y(t))dt \tag{2}$$

and conversely.

Using the above lemma, we present the proof of the main result in detail.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1) in $|x - x_0| \le a$. We will show that $y(x) = \bar{y}(x)$ in the interval $[x_0, x_0 + a]$.

We now let $\emptyset(x) = |y(x) - \bar{y}(x)|$. From the assumption (4) and (2), we obtain

$$\begin{aligned}
\emptyset(x) &\le \int_{x_0}^{x} |f(t,y(t)) - f(t,\bar{y}(t))|dt \\
&\le \int_{x_0}^{x} \frac{m}{(x-x_0)^\gamma}|y(t) - \bar{y}(t)|^\alpha dt \\
&= \int_{x_0}^{x} \frac{m}{(x-x_0)^\gamma}\emptyset^\alpha(t)dt
\end{aligned}$$

We get

$$R(x) = \int_{x_0}^{x} \frac{m}{(x-x_0)^\gamma}\emptyset^\alpha(t)dt,$$

then

$$R(x_0) = 0,$$

and

$$R'(x) = \frac{m}{(x - x_0)^\gamma} \emptyset^\alpha(x)$$
$$\leq \frac{m}{(x - x_0)^\gamma} R^\alpha(x)$$

This implies that

$$R'(x) - \frac{m}{(x - x_0)^\gamma} R^\alpha(x) \leq 0$$

Since $R(x) > 0$ for all $x > x_0$, on multiplying this inequality by $(1 - \alpha)R^{-\alpha}(x)$, we have

$$(R^{1-\alpha}(x))' \leq \frac{m(1 - \alpha)}{(x - x_0)^\gamma}$$
$$\leq \frac{m}{(x - x_0)^\gamma}$$

and hence

$$R^{1-\alpha}(x) \leq \int_{x_0}^{x} \frac{m}{(t - x_0)^\gamma} dt$$
$$= \frac{m}{(1 - \gamma)(x - x_0)^{\gamma - 1}}$$

Thus, it follows that

$$\emptyset(x) \leq R(x)$$
$$\leq \left(\frac{m}{1 - \gamma}(x - x_0)^{\gamma - 1}\right)^{(1-\alpha)^{-1}}$$

So, the function $\omega(x) = \frac{\emptyset(x)}{(x - x_0)^k}$ satisfies the inequality

$$0 \leq \omega(x) \leq \left(\frac{m}{1 - \gamma}\right)^{(1-\alpha)^{-1}} (x - x_0)^{(1-\gamma)(1-\alpha)^{-1} - k}$$

Since $\gamma + k(1 - \alpha) < 1$, it leads to $\lim_{x \to {x_0}^+} \omega(x) = 0$. Therefore, if we define $\omega(x) = 0$,

then the function $\omega(x)$ is continuous in $[x_0, x_0 + a]$. We will show that $\omega(x) = 0$ in $[x_0, x_0 + a]$. In fact, if $\omega(x) > 0$ at any point in $[x_0, x_0 + a]$, then there exists a point $x_1 > x_0$ such that

$$0 < l = \omega(x_1) = max_{x_0 \leq x \leq x_0 + a} \omega(x)$$

However, from (3) we obtain

$$l = \omega(x_1) = \frac{\emptyset(x_1)}{(x_1 - x_0)^k}$$
$$\leq (x_1 - x_0)^{-k} \int_{x_0}^{x_1} |f(t, y(t)) - f(t, \bar{y}(t))| dt$$

$$\leq (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k|t - x_0|^{-1} |y(t) - \bar{y}(t)|\, dt$$

$$\leq (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{-1} \emptyset(t)\, dt$$

$$\leq (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{-1} (t - x_0)^k \omega(t) dt$$

$$< l(x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{k-1} dt$$

$$= \ l,$$

which is the desired contradiction. Thus, $\omega(x) = 0$, and then $\emptyset(x) = 0$ in $[x_0, x_0 + a]$.

The proof is similar in the interval $[x_0 - a, x_0]$. ■

4.2. Example

In this section, we construct an example to show the applicability of the obtained theoretical result.

More specifically, we consider the initial value problem:

$$y' = f(x,y) = \begin{cases} 0, & 0 \leq x \leq 1,\ (x - x_0)^{(1-\alpha)^{-1}} < y < \infty \\ k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} - k\frac{y}{(x-x_0)^{\gamma}}, & 0 \leq x \leq 1, x \neq x_0, 0 \leq y \leq (x - x_0)^{(1-\alpha)^{-1}} \\ k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}}, & 0 \leq x \leq 1, x \neq x_0, -\infty < y < 0 \end{cases} \quad (5)$$

$y(0)=0$,

where $0<\alpha < 1, k > 0$ and $\gamma < 1,\ \gamma + k(1 - \alpha) < 1$.

Solve. This function $f(x,y)$ is continuous at $(0, y)$ for $0 \leq y \leq (x - x_0)^{(1-\alpha)^{-1}}$ since

$$\left|k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} - k\frac{y}{(x - x_0)^{\gamma}}\right|$$

$$\leq k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} + k\frac{(x - x_0)^{(1-\alpha)^{-1}}}{(x - x_0)^{\gamma}}$$

$$= 2k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} \to 0 \text{ as } x \to x_0$$

Thus, it is clear that $f(x,y)$ is continuous in the strip $0 \leq x \leq 1, |y| < \infty$.

Next, we shall verify the conditions (3) and (4) by considering the following cases:

Suppose $0 \leq y, \bar{y} \leq (x - x_0)^{(1-\alpha)^{-1}}$ *then*

$$|f(x,y) - f(x,\bar{y})| = \left|-k\frac{y}{(x - x_0)^{\gamma}} + k\frac{\bar{y}}{(x - x_0)^{\gamma}}\right|$$

$$= \frac{k}{(x - x_0)^{\gamma}} |y - \bar{y}|$$

$$\leq \frac{k}{x - x_0}|y - \bar{y}|$$

and

$$|f(x,y) - f(x,\bar{y})| = \frac{k}{(x - x_0)^{\gamma}}|y - \bar{y}|^{1-\alpha}|y - \bar{y}|^{\alpha}$$

$$\leq \frac{k}{(x - x_0)^{\gamma}}(|y| + |\bar{y}|)^{1-\alpha}|y - \bar{y}|^{\alpha}$$

$$\leq \frac{k}{(x - x_0)^{\gamma}}2^{1-\alpha}(x - x_0)^{(1-\alpha)^{-1}(1-\alpha)}|y - \bar{y}|^{\alpha}$$

$$= \frac{2^{1-\alpha}k}{(x - x_0)^{\gamma-1}}|y - \bar{y}|^{\alpha}.$$

Suppose $(x - x_0)^{(1-\alpha)^{-1}} < y < \infty, -\infty < \bar{y} < 0,$ then

$$|f(x,y) - f(x,\bar{y})| = \left|-k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}}\right|$$

$$< \frac{k}{(x - x_0)^{\gamma}}y$$

$$< \frac{k}{(x - x_0)^{\gamma}}|y - \bar{y}|$$

$$< \frac{k}{x - x_0}|y - \bar{y}|$$

and

$$|f(x,y) - f(x,\bar{y})| = k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}}$$

$$= k(x - x_0)^{(1-\gamma)}(x - x_0)^{\alpha(1-\alpha)^{-1}}$$

$$< \frac{k}{(x - x_0)^{(\gamma-1)}}y^{\alpha}$$

$$< \frac{k}{(x - x_0)^{(\gamma-1)}}|y - \bar{y}|^{\alpha}$$

$$< \frac{2^{1-\alpha}k}{(x - x_0)^{\gamma}}|y - \bar{y}|^{\alpha}$$

Suppose $(x - x_0)^{(1-\alpha)^{-1}} < y < \infty, 0 \leq \bar{y} \leq (x - x_0)^{(1-\alpha)^{-1}}$ then

$$|f(x,y) - f(x,\bar{y})| = \left|-k(x - x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} + k\frac{\bar{y}}{(x-x_0)^{\gamma}}\right|$$

$$= \frac{k}{(x - x_0)^{\gamma}}\left|(x - x_0)^{(1-\alpha)^{-1}} - \bar{y}\right|$$

$$< \frac{k}{x - x_0}|y - \bar{y}|$$

and

$$|f(x,y)-f(x,\bar{y})| = k\left|\frac{(x-x_0)^{(1-\alpha)^{-1}}-\bar{y}}{(x-x_0)^{\gamma}}\right|$$

$$= \frac{k}{(x-x_0)^{\gamma-1}}\left[(x-x_0)^{(1-\alpha)^{-1}}-\bar{y}\right]^{\alpha}\left[\frac{(x-x_0)^{(1-\alpha)^{-1}}-\bar{y}}{(x-x_0)^{(1-\alpha)^{-1}}}\right]^{1-\alpha}$$
$$\leq \frac{k}{(x-x_0)^{\gamma-1}}\left[(x-x_0)^{(1-\alpha)^{-1}}-\bar{y}\right]^{\alpha}$$
$$< \frac{k}{(x-x_0)^{\gamma-1}}[y-\bar{y}]^{\alpha}$$
$$< \frac{2^{1-\alpha}k}{(x-x_0)^{\gamma}}|y-\bar{y}|^{\alpha}$$

Suppose $0 \leq y \leq (x-x_0)^{(1-\alpha)^{-1}}, -\infty < \bar{y} < 0$, then

$$|f(x,y)-f(x,\bar{y})|$$
$$= \left|k(x-x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}} - k\frac{y}{(x-x_0)^{\gamma}}\right.$$
$$\left. - k(x-x_0)^{(1-\gamma+\alpha\gamma)(1-\alpha)^{-1}}\right| = \left|k\frac{y}{(x-x_0)^{\gamma}}\right| < \frac{k}{(x-x_0)^{\gamma}}|y-\bar{y}|$$
$$< \frac{k}{x-x_0}|y-\bar{y}|$$

and

$$|f(x,y)-f(x,\bar{y})| = k\frac{y}{(x-x_0)^{\gamma}}$$
$$\leq \frac{ky^{\alpha}}{(x-x_0)^{(\gamma-1)}}$$
$$< \frac{k|y-\bar{y}|^{\alpha}}{(x-x_0)^{(\gamma-1)}}$$
$$< \frac{2^{1-\alpha}k|y-\bar{y}|^{\alpha}}{(x-x_0)^{\gamma}}$$

Since all the conditions of theorem 3.1 are satisfied, the initial value problem (5) has a unique solution in [0, 1].

5. Conclusions

In this paper, we have extended the uniqueness result in the work of Krasnosel'skii and Krein (Krasnoselskii and Krein, 1956). The obtained result can apply to many classes of differential equations where the source functions have a singularity. An example has been constructed to show the applicable of the theoretical result. We emphasize that the results in (Krasnoselskii and Krein, 1956) cannot give us any conclusion on the uniqueness of the equation in this example. In future works, we will develop our method to study the other class of differential equations with singularity source terms.

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