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A Lyapunov-type inequality for a fractional differential equation under multi-point boundary conditions

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ABSTRACT

In this paper we consider the value boundary problem

$$\begin{cases} {}_{a+}^{C}D^{\alpha,g}\mathrm{y(t)} + \mathrm{q(t)y(t)} = 0, \mathrm{a} < \mathrm{t} < \mathrm{b}, 1 < \alpha \le 2, \\ y(a) = y(b) = 0, \end{cases}$$

where $g \in C_+^1[a,b]$, *and* $q: [a,b] \to R$ *is a continuous function. We obtained a Lyapunov-type inequality as follows:*

$$\int_a^b [g(b) - g(s)]^{\alpha-1} g'(s)|q(s)|ds \ge \Gamma(\alpha)$$

This result is new to the corresponding results in the literature.

Keywords: *Lyapunov-type inequalities, the generalized Caputo fractional derivatives, the Green's function*

1. Introduction

If *y(t)* is a nontrivial solution of differential system

$$\begin{cases} \mathrm{y''(t)} + \mathrm{r(t)y(t)} = 0, \mathrm{a} < \mathrm{t} < \mathrm{b}, \\ y(a) = y(b) = 0, \end{cases}$$

where *r(t)* is a continuous function defined in *[a,b]*, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a} \quad \text{(Lyapunov, 1907)}.$$

Lyapunov-type inequalities for fractional differential equations with different boundary conditions have been investigated by many researchers in recent years.

Ferreira (2013) considered the fractional differential equation with boundary conditions:

$$\begin{cases} D^{\alpha}\mathrm{y}(\mathrm{t})+\mathrm{q}(\mathrm{t})\mathrm{y}(\mathrm{t})=0, \mathrm{a}<\mathrm{t}<\mathrm{b}, 1<\alpha \le 2, \\ y(a)=y(b)=0 \end{cases} \tag{1.1}$$

where $D^{\alpha}(.)$ is the Riemann-Liouville fractional derivative, and $q:[a,b]\rightarrow R$ is a continuous function. He obtained a Lyapunov-type inequality for the problem (1.1) as follows:

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}$$

Ferreira (2014) replaced the Reimann-Liouville fractional derivative in problem (1.1) with Caputo fractional derivative ${}_{a}^{C}D^{\alpha}(.)$:

$$\begin{cases} {}_{a}^{C}D^{\alpha}\mathrm{y}(\mathrm{t})+\mathrm{q}(\mathrm{t})\mathrm{y}(\mathrm{t})=0, \mathrm{a}<\mathrm{t}<\mathrm{b}, 1<\alpha \le 2, \\ y(a)=y(b)=0, \end{cases} \tag{1.2}$$

and he obtained a Lyapunov-type inequality for the problem (1.2) as follows:

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}} \tag{1.3}$$

In this paper, we replace the Caputo fractional derivative in problem (1.2) with the left g-Caputo fractional derivative ${}_{a+}^{C}D^{\alpha,g}(.)$. Particularly, we consider the boundary value problem:

$$\begin{cases} {}_{a+}^{C}D^{\alpha,g}\mathrm{y}(\mathrm{t})+\mathrm{q}(\mathrm{t})\mathrm{y}(\mathrm{t})=0, \mathrm{a}<\mathrm{t}<\mathrm{b}, 1<\alpha \le 2, \\ y(a)=y(b)=0 \end{cases} \tag{1.4}$$

where $g \in C_{+}^{1}[a,b]$, and $q:[a,b]\rightarrow R$ is a continuous function.

We obtained a Lyapunov-type inequality for the problem (1.4) as follows:

$$\int_a^b [g(b)-g(s)]^{\alpha-1}g'(s)|q(s)|ds \ge \Gamma(\alpha) \tag{1.5}$$

This result is new to the corresponding results in the literature.

As a special case (see Corollary 3.4), letting *g(t)=t, t ∈ [a,b]* in the problem (1.4) reduces it to the problem (1.2) and the corresponding inequality becomes

$$\int_a^b (b-s)^{\alpha-1}|q(s)|ds \ge \Gamma(\alpha)$$

We give an example (see Example 3.5) in which we use Corollary 3.4 to show that the boundary value problem has no nontrivial solution.

2. Preliminaries

In this section, we recall some basic definitions. For convenience in writing, we denote

$$C_+^1[a,b] = \{g \in C^1[a,b]: g'(t) > 0, \forall\, t \in [a,b]\}$$

Definition 2.1 (I. Podlubny, 1999). *Let $\phi \in C^n[a,b], n \in N^+$, and $\alpha \in (n, n\text{-}1)$, then the Caputo fractional derivative of order α is the expression*

$$ {}_a^C D^\alpha \phi(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \phi(s) \mathrm{d}s,$$

where $\Gamma(.)$ is the Gamma function.

Ddefinition 2.2 (T.J. Osler, 1970). *Let $\alpha > 0$, $g \in C_+^1[a,b]$, and $\phi \in C^1[a,b]$. The fractional integral of a function ϕ with respect to the function g is defined by*

$$I_{a+}^{\alpha,g} \phi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t [g(t) - g(s)]^{\alpha-1} g'(s) \phi(s) \mathrm{d}s.$$

Definition 2.3 (R. Almeida, 2017). *Let $\alpha > 0$, $n \in N^+$; $g, \phi \in C^n[a,b]$ two functions such that $g'(t)>0$, $\forall t \in [a, b]$. The left g-Caputo fractional derivative of ϕ of order α is given by*

$$ {}_{a+}^{C}D^{\alpha,g}\phi(t) = I_{a+}^{n-\alpha,g}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_a^t [g(t)-g(s)]^{n-\alpha-1} g'(s) \left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n \phi(s)\mathrm{d}s.$$

For $g(t)=t$, $t \in [a,b]$, the left g-Caputo fractional derivative ${}_{a+}^{C}D^{\alpha,g}(.)$ is becomes the Caputo fractional derivative ${}_a^C D^\alpha(.)$.

Lemma 2.4 (R. Almeida, 2017). *Let $n \in N^+, n-1 < \alpha < n$, and $g \in C_+^1[a,b]$, we have*

$$\left(I_{a+}^{\alpha,g}\, {}_{a+}^{C}D^{\alpha,g}\phi\right)(t) = \phi(t) + \sum_{k=0}^{n-1} c_k[g(t)-g(a)]^k,\ c_k \in R, (k = 0, \dots, n-1).$$

3. Main Results

Lemma 3.1. *Let $1 < \alpha \leq 2$, and $g \in C_+^1[a,b]$. Suppose that y(t) is a solution of the problem (1.4). Then y(t) is a solution of the following integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s)[g(b)-g(s)]^{\alpha-1} g'(s) q(s) y(s) \mathrm{d}s,$$

where

$$G(t,s)=\begin{cases}\dfrac{g(t)-g(a)}{g(b)-g(a)}-\left(\dfrac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha-1}, & a\le s<t\le b,\\ \dfrac{g(t)-g(a)}{g(b)-g(a)}, & a\le t\le s\le b.\end{cases} \quad (3.1)$$

Proof. By using Lemma 2.4, we can rewrite (1.4) in the following form

$$y(t)=-I_{a+}^{\alpha,g}y(t)q(t)+\ c_0+c_1[g(t)-g(a)],(c_0,c_1\in R)$$
$$=\ \frac{-1}{\Gamma(\alpha)}\int_a^t[g(t)-g(s)]^{\alpha-1}g'(s)q(s)y(s)\mathrm{d}s+c_0+c_1[g(t)-g(a)]$$

From the condition *y(a)=0*, we see that $c_0=0$. Furthermore, from *y(b)=0*, we get

$$c_1=\ \frac{1}{[g(b)-g(a)]\Gamma(\alpha)}\int_a^b[g(b)-g(s)]^{\alpha-1}g'(s)q(s)y(s)\mathrm{d}s.$$

Thus, we obtain

$$y(t)=\ \frac{1}{\Gamma(\alpha)}\int_a^b G(t,s)[g(b)-g(s)]^{\alpha-1}g'(s)q(s)y(s)\mathrm{d}s,$$

where

$$G(t,s)=\begin{cases}\dfrac{g(t)-g(a)}{g(b)-g(a)}-\left(\dfrac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha-1}, & a\le s<t\le b,\\ \dfrac{g(t)-g(a)}{g(b)-g(a)}, & a\le t\le s\le b.\end{cases}$$

The proof of Lemma is completed.

Lemma 3.2. *Let the Green's function G(t,s) be defined as in (3.1). Then*

$$\max_{t,s\in[a,b]}|G(t,s)|=1.$$

Moreover,

$|G(t,s)|=1$ *if and only if t=s= b.*

Proof. For $a\le t\le s\le b$, we have $G(t,s)=\frac{g(t)-g(a)}{g(b)-g(a)}$. Clearly,

$0\le G(t,s)\le 1$, and *G(t,s)=1* if and only if *t=s= b.*

For $a\le s<t\le b$, we consider the function

$$h(t,s)=\frac{g(t)-g(a)}{g(b)-g(a)}-\left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha-1},a\le s<t\le b,1<\alpha\le 2.$$

By fixing $t\in[a,b]$ and taking the derivative with respect to *s*, we get

$$\frac{\partial h}{\partial s}(t,s) = (\alpha-1)\frac{g'(s)[g(b)-g(t)]}{[g(b)-g(s)]^2}\left(\frac{g(t)-g(s)}{g(b)-g(s)}\right)^{\alpha-2} \geq 0, \forall\, a \leq s < t \leq b.$$

Hence, *h(t,s)* is a monotone function of *s*, so

$1 > h(t,t) > h(t,s) \geq h(t,a)$, for a ≤ s < t ≤ b. (3.3)

On the other hand,

$$h(t,a) = \frac{g(t)-g(a)}{g(b)-g(a)} - \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha-1} < 0, \forall t \neq a. \quad (3.4)$$

Combining (3.2), (3.3), and 3.4), we get

$$\max_{t,s\in[a,b]} |G(t,s)| = \max_{t\in[a,b]}\{1, |h(t,a)|\}.$$

By differentiating *h(t,a)* with respect to *t*, we obtain

$$\frac{\partial h}{\partial t}(t,a) = \frac{g'(t)}{g(b)-g(a)}\left[1-(\alpha-1)\left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha-2}\right].$$

Thus

$$\frac{\partial h}{\partial t}(t^*,a) = 0 \Leftrightarrow g(t^*) = (\alpha-1)^{\frac{1}{2-\alpha}}[g(b)-g(a)] + g(a).$$

Since $g(a) < g(t^*) < g(b)$ and $g'(t) > 0, \forall t \in [a,b]$, we have $a < t^* < b$.

Note that, *h(a,a)=h(b,a)=0*, and

$$h(t^*,a) = \alpha^{\frac{1}{2-\alpha}}\,[1-(\alpha-1)^{\alpha-1}] < 0,$$

we can conclude that

$$\max_{t\in[a,b]} |h(t,a)| = |h(t^*,a)| = \alpha^{\frac{1}{2-\alpha}}\,[(\alpha-1)^{\alpha-1}-1] < 1.$$

In the case of $\alpha=2$, then $h(t,a)=0,\ \forall t \in [a,b]$.

Hence,

$\max_{t,s\in[a,b]} |G(t,s)|$=max $\{1, |h(t^*,a)|\} = 1$ and $|G(t,s)|=1$ if and only if $t=s=b$,

as required.

Theorem 3.3. *Suppose that y(t) is the nontrivial solution of the problem (1.4), then*

$$\int_a^b [g(b)-g(a)]g'(s)|q(s)|ds \geq \Gamma(\alpha).$$

Proof. By Lemma 3.1, we have

$$|y(t)| \leq \frac{1}{\Gamma(\alpha)}\int_a^b |G(t,s)|[g(b)-g(s)]^{\alpha-1}g'(s)|q(s)||y(s)|\mathrm{d}s, \forall t \in [a,b],$$

$$\leq \frac{||y||}{\Gamma(\alpha)}\int_a^b [g(b)-g(s)]^{\alpha-1}g'(s)|q(s)|\mathrm{d}s.$$

Hence,

$$||y|| \leq \frac{||y||}{\Gamma(\alpha)}\int_a^b [g(b)-g(s)]^{\alpha-1}g'(s)|q(s)|\mathrm{d}s,$$

or

$$\int_a^b [g(b)-g(s)]^{\alpha-1}g'(s)|q(s)|\mathrm{d}s \geq \Gamma(\alpha),$$

which finishes the proof.

When $g(t)=t,\ t \in [a,b]$, then the problem (1.4) reduces it to the problem (1.2). From Theorem 3.3 we get the following result:

Corollary 3.4. *If*

$$\int_a^b (b-s)^{\alpha-1}|q(s)|\mathrm{d}s < \Gamma(\alpha),$$

then the boundary value problem (1.2) has no nontrivial solution.

Example 3.5. *Consider the boundary value problem:*

$$\begin{cases} {}_{0^+}^{C}D^{1.5}y(t) + \frac{5t}{2}y(t) = 0, 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \tag{3.5}$$

Since

$$\frac{5}{2}\int_0^1 (1-s)^{0.5}s\mathrm{d}s = \frac{2}{3} < \Gamma(1.5) \approx 0.88623,$$

we see that the problem (3.5) has no nontrivial solution, by Corollary 3.4.

We apply the Theorem 3.3 to find the bound for the eigenvalue of the fractional boundary value problem:

Corollary 3.6. *If the fractional boundary value problem*

$$\begin{cases} {}_{a^+}^{C}D^{\alpha,g}y(t) + \lambda y(t) = 0, a < t < b, 1 < \alpha \leq 2, \\ y(0) = y(1) = 0, \end{cases}$$

has a nontrivial solution, then

$$\int_a^b [g(b)-g(s)]^{\alpha-1}g'(s)|\lambda|\mathrm{d}s \geq \Gamma(\alpha).$$

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