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Comparison of Synchronization Speed of Regular Networks of Reaction-Diffusion Systems of FitzHugh-Nagumo Type with Unidirectional and Bidirectional Coupling

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ABSTRACT

In this study, the author researches the numerical results of the comparison of synchronization speed of regular neural networks with unidirectional and bidirectional coupling. Each neuron is linearly coupled with the others and is represented by a reaction-diffusion system of FitzHugh-Nagumo type. The result shows that the necessary coupling strength for the synchronization in two cases decreases when the coupling number of neurons increases. In other words, the bigger the coupling number of neurons in the regular networks is, the easier the synchronization occurs. Moreover, synchronizing the regular networks with bidirectional coupling is easier than the one with unidirectional coupling over the same given number of neurons.

Keywords: *reaction-diffusion system of FitzHugh-Nagumo, synchronization, coupling strength, regular network, unidirectional coupling, bidirectional coupling*

1. Introduction

Synchronization is an extremely important phenomenon in nature and nonlinear science, especially in the network of interconnected dynamical systems (Aziz-Alaoui, 2006; Keener & Sneyd, 2009; Murray, 2010). That means the systems will have the same behavior at the same time. Specifically, for a network of two systems, synchronization means that one system will copy the properties of the other from a certain time. Then, the network is said to be synchronous.

In the human brain, many cells connect to form a network of cells. A cellular network is a system of cells that are physiologically linked together. The exchange between them is mainly based on electrochemical processes. This paper numerically presents the synchronization of a system of linearly interconnected cells. In which, each cell is described by a system of reaction-diffusion equations of FitzHugh-Nagumo type (FHN).

In 1952, Hodgkin and Huxley proposed a four-dimensional mathematical model that could approximate the energizing properties of cell voltage (Hodgkin & Huxley, 1952; Corson, 2009). Based on this model, many simpler models have been published to describe the cell voltage dynamics. In 1962, FitzHugh and Nagumo published a new model named FitzHugh-Nagumo model known as a simplified two-dimensional model from the famous system of Hodgkin-Huxley (Hodgkin & Huxley, 1952; Ambrosio & Aziz-Alaoui, 2013; Izhikevich, 2007). Although the model is simpler, it has many remarkable analytical results and retains the properties and biological significance. This model is made up of two equations of two variables u and v. The first variable is the fast variable, called the active variable, which represents the voltage of the cell membrane. The second variable is the slow variable, which represents some time-dependent physical quantity such as the conductivity of the flow of ions across the cell membrane.

The FitzHugh-Nagumo model is represented by the following system, using the notation as in (Ambrosio & Aziz-Alaoui, 2012; Ambrosio & Aziz-Alaoui, 2013):

$$\begin{cases} \varepsilon \dfrac{du}{dt} = f(u) - v, \\ \dfrac{dv}{dt} = au - bv + c, \end{cases} \tag{1}$$

where a, b and c are constant (a and b are positive), $0 < \varepsilon \ll 1$ and $f(u) = -u^3 + 3u$. However, this system is not strong enough to reflect the propagation of cell voltage in space (along the cell body), so the cable equations are used here by adding the Laplace operator to the system (1) as follows:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = \varepsilon u_t = f(u) - v + d\Delta u, \\ \dfrac{dv}{dt} = v_t = au - bv + c, \end{cases} \tag{2}$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d$ is positive constant, Δu is Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a uniformly bounded open set and the system satisfies the Neumann zero flux condition on the boundary (N is a positive integer). This system consists of two parabolic nonlinear partial differential equations, allowing a wide variety of physiologically voltage-related and diverse shapes of cell membranes to be expressed (Ambrosio & Aziz-Alaoui, 2012; Ambrosio & Aziz-Alaoui, 2013). Note that the first equation, also known as the cable equation, describes the flow of potentials along the body of a cell (Ermentrout & Terman, 2009; Izhikevich, 2007).

The system (2) is considered as a model of a cell from which a cell network consisting of *n* systems (2) is linked together by the following system:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i + d\Delta u_{u_i} - h(u_i, u_j) \\ v_{it} = au_i - bv_i + c \end{cases} \quad i, j = 1,...,n, i \neq j, \qquad (3)$$

where $(u_i, v_i), i = 1, 2,..., n$ is defined as the system (2).

Function *h* is the coupling function describing the type of association between cells *i*th and *j*th. There are two types of connections between cells: chemical and electrical. This study only focuses on electrical connection, then the coupling function is linear and is given by the following formula:

$$h(u_i, u_j) = g_n \sum_{j=1}^{n} c_{ij}(u_i - u_j), \quad i = 1, 2,..., n. \qquad (4)$$

The parameter g_n is a constant describing the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$ defined by: if u_i and u_j are coupled, $c_{ij} = 0$ if u_i and u_j are not coupled, where $i, j = 1, 2,..., n, i \neq j.$

In recent years, synchronization has been widely studied in many fields, many natural phenomena also reflect synchronization such as the movement of birds forming clouds, the movement of carps in a lake, the movement of a parade, the reception and transmission of information by a group of cells, etc. (Nagumo, Arimoto & Yoshizawa, 1962; Keener & Sneyd, 2009; Murray, 2010). Therefore, the study of synchronization in the network of cells is essential. In this paper, the author presents numerical results on comparing the synchronization speed of regular networks of FHN with unidirectional and bidirectional coupling.

2. Synchronization speed of regular networks of reaction-diffusion systems of FitzHugh-Nagumo type with unidirectional and bidirectional coupling

A cellular network is a physiologically interconnected system of cells. The exchange between them is mainly based on electrochemical processes, each element (node) of the network is a cell modeled by a system of reaction-diffusion equations of FHN and each edge represents a cell junction simulated by the coupling function (Aziz-Alaoui, 2006; Ambrosio & Aziz-Alaoui, 2012; Ambrosio & Aziz-Alaoui, 2013). To make the study easy, this paper only considers the connection between cells in electrical type, that is, the cells are linked linearly with each other. Because of the interconnectedness of a network of many cells, there will have a moment when synchronization will occur with some corresponding conditions. To make it easier to imagine, synchronization for a network of two systems means that one system will copy the properties of the other from a certain time (Definition 1). The synchronization of the network of FHN in this paper is performed on a regular structure with bidirectional and unidirectional coupling.

Figure 1 below is an example of a regular network of 10 FHN cells with unidirectional coupling, shown by arrows, meaning that the i th cell will couple with the $(i+1)$th cell $(i=1,2,...,9)$, and the tenth cell will couple with the first cell. Next, the i th cell will couple with the ($i+1$)th and ($i+2$)th cell ($i=1,2,...,8$), and the tenth cell will associate with the first and second cell. Similarly, this type of connection causes the number of input links in each cell to increase gradually from 1 to 8 (the illustration shows the number of input links in each cell increasing from 1 to 4).

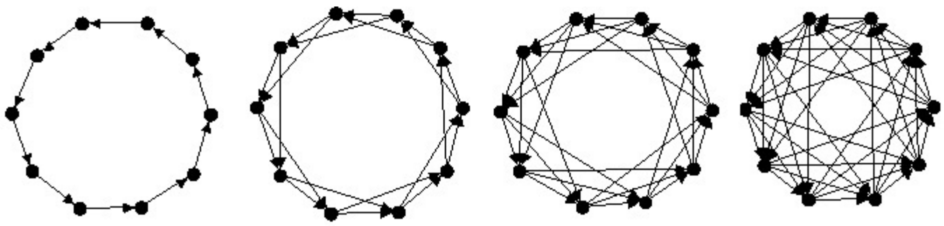

Figure 1. The graphs consist of 10 cells linearly linked together, the number of input links in each cell increases from 1 to 4

Figure 2 is an example of a regular network of 20 FHN cells with bidirectional coupling, meaning that the first cell will associate with the second and twentieth cell; and the ith cell will associate with the ($i+1$)th and ($i-1$)th cell ($i=2,3,...,19$). Finally, the twentieth cell will associate with the first and nineteenth cell. Next, the first cell will associate with the second, third, nineteenth, and twentieth cell; the second cell will associate with the third, fourth, first and twentieth cell; and the ith cell associates with the $(i+1)$th, ($i+2$)th, $(i-1)$th and $(i-2)$th cell $(i=3,4,...,19)$. Similarly, this type of connection causes the number of input links in each cell to increase from 2,4,6,…, to 18.

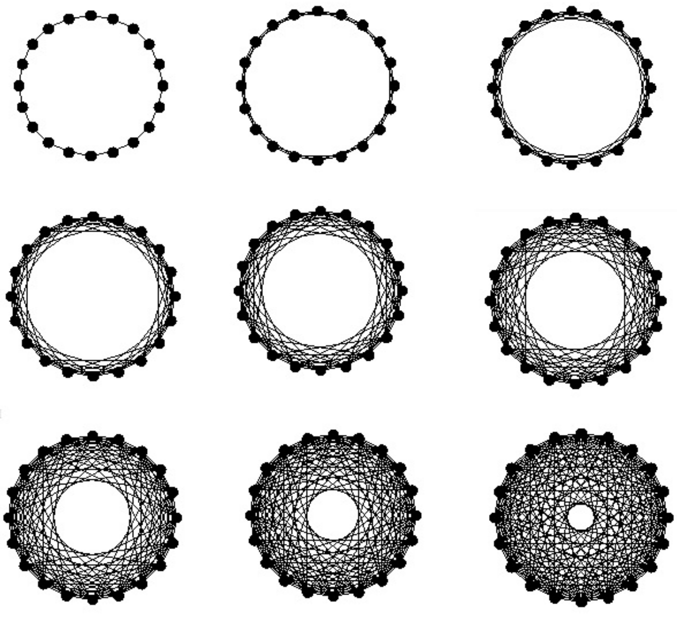

Figure 2. The network consists of 20 cells linearly linked together, the number of input links in each cell increases from 2,4,6,…, to 18

Definition 1. *Let* $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ *and* $S = (S_1, S_2, ..., S_n)$ *be a network. We say that* S *is identically synchronous if*

$$\lim_{t\to+\infty}\sum_{i=1}^{n-1}\left(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)}\right) = 0,$$

where $L^2(\Omega)$ is function space on Ω defined using a natural generalization of the *2*-norm for finite-dimensional vector spaces.

3. Simulation results

In this section, the results of the paper are performed by numerical methods for the regular network of 20 systems (2) that are linearly linked together by the unidirectional way and bidirectional one, where $f(u) = -u^3 + 3u,\ a = 1, b = 0.001, c = 0, \varepsilon = 0.1, d = 0.05, i = 1, 2, 3.$ This numerical method is performed on C++, với $[0;T]\times\Omega = [0;200]\times[0;100]\times[0;100].$ Specifically, let d_{in} be the number of input links of each node of the network, each time we increase the number of input links, we perform numerical methods to find the small enough coupling strength needed to obtain the synchronization. Note that, for the regular network with unidirectional coupling, the number of input links at each node only increases by 1 each time, while for the regular network with bidirectional coupling, the number of input links will increase double after each execution.

In order to better understand how the synchronization happens, Figure 3 below is the result depicting the synchronization phenomenon in the regular network of 3 systems of FHN with unidirectional coupling. The results show that the synchronization is obtained since the value $g_3 = 0.04.$

Figure 3(a), 3(b), 3(f), 3(g), 3(k), 3(l), 3(p), 3(q) represent the synchronization errors of the coupled solutions $\left(u_1(x_1, x_2, t), u_2(x_1, x_2, t)\right)$ and $\left(u_2(x_1, x_2, t), u_3(x_1, x_2, t)\right)$, where $t \in [0;T]$ and for all $(x_1, x_2) \in \Omega$.

In Figure 3(p) and 3(q) with $g_3 = 0.04,$ the simulation shows that the synchronization errors reach zero, it means:

$$u_1(x_1, x_2, t) \approx u_2(x_1, x_2, t) \text{ and } u_2(x_1, x_2, t) \approx u_3(x_1, x_2, t)$$

for all $(x_1, x_2) \in \Omega$.

Figure 3(c), 3(d), 3(e), 3(h), 3(i), 3(j), 3(m), 3(n), 3(o), 3(r), 3(s), 3(t) represent the solutions $u_i(x_1, x_2, 190), i = 1, 2, 3,$ of the network from when no synchronization has occurred until they have the same shape, i.e, the synchronization is performed.

Before synchronization with $g_3 = 0.01$, Figure 3(a) represents the synchronization error between u_2 and u_1, for all $(x_1, x_2) \in \Omega$; Figure 3(b) represents the synchronization error between u_3 and u_2; Figure 3(c) represents a solution $u_1(x_1, x_2, 190)$; similarly, Figure 3(d) and 3(e) represent the solutions $u_2(x_1, x_2, 190)$ and $u_3(x_1, x_2, 190)$ when they are coupled together; the results are similarly done for $g_3 = 0.02$ (Figure 3(f), 3(g), 3(h), 3(i), 3(j)), $g_3 = 0.03$ (Figure 3(k), 3(l), 3(m), 3(n), 3(o)) and $g_3 = 0.04$ (Figure 3(p), 3(q), 3(r), 3(s), 3(t)). For $g_3 = 0.04$, the synchronization occurs, since the synchronization errors reach zero (see Figure 3(p), 3(q)).

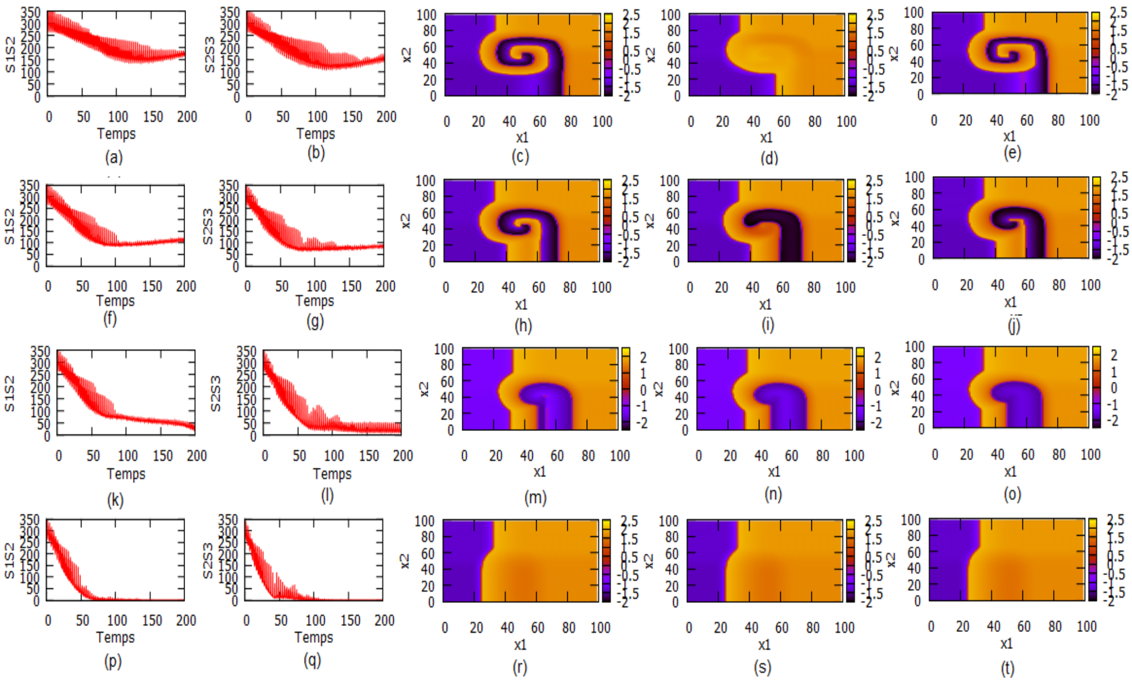

Figure 3. Synchronization in the regular network of 3 systems of FHN with unidirectional coupling

From the above result, in the case of three linearly coupled neurons, the coupling strength over or equal to $g_3 = 0,04$, these neurons has synchronous behaviors. By doing similarly for the regular networks of 20 systems of FHN with unidirectional coupling, we have Table 1 below reporting the values of coupling strength according to the number of input links of each node (d_{in}) that gradually increase from 1 to 9.

TABLE 1. Minimal coupling strength necessary to observe the synchronization in the regular network with unidirectional coupling

Input links d_{in}	1	2	3	4	5	6	7	8	9
g_{20}	0.03	0.02	0.013	0.011	0.0088	0.008	0.0068	0.0064	0.006

Following these numerical experiments, it is easy to see that the coupling strength required to observe the synchronization in the regular networks of FHN with unidirectional coupling depends on the number of input links of each neuron. Indeed, the points in Figure 4(a) represent the values of coupling strength according to the number of input links of each node in the regular network from Table 1, and we find a function presenting the relation between the number of input links of each node and the coupling strength reported in Table 1. This function is as follows:

$$g_{20} = \frac{0.0265}{d_{in}} + 0.0035, \ (5)$$

where, d_{in} is the number of input links of each node in the regular network. In Figure 4(a), function (5) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means that the coupling strength necessary to obtain the synchronization in the regular network follows the law presented by (5). These simulations show that the bigger the number of input links of each neuron is, the smaller the coupling strength is. It means that synchronization is easier when d_{in} in the regular network is bigger.

Similarly, for the regular network of FHN with bidirectional coupling, the results in Table 2 below show the change of coupling strength corresponding to the increasing of input links of each node from 2,4,6,…,18.

TABLE 2. Minimal coupling strength necessary to observe the synchronization in the regular network with bidirectional coupling

Input links d_{in}	2	4	6	8	10	12	14	16	18
g_{20}	0.02	0.012	0.008	0.0065	0.005	0.004	0.0037	0.0035	0.003

In Figure 4(b), the points represent the values of coupling strength according to the number of input links of each node in the regular network from Table 2, and we find a function presenting the relation between the number of input links of each node and the coupling strength reported in Table 2. This function is as follows:

$$g_n = \frac{0.0375}{d_{in}} + 0.00125, \ (6)$$

where, d_{in} is the number of input links of each node in the regular network. In Figure 4(b), function (6) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means that the coupling strength necessary to obtain the synchronization in the regular network follows the law presented by (6). These simulations also show that synchronization is easier when d_{in} in the regular networks is bigger.

Based on Figure 4, it can be seen that the coupling strength required for the synchronization in the regular network of FHN in both cases tends to decrease as the number of input links at each node is increased. That is, as the number of input links at each node in the network increases, synchronization is more likely to occur. Moreover, the comparison between the two results in Figures 4(a) and 4(b) shows that the synchronization speed of the regular network with bidirectional coupling will be faster than the regular network with unidirectional coupling on the same number of neurons. This is completely consistent with the law of nature, when the more links and information exchanged between members in the network, the easier it will be to have mutual synchronization.

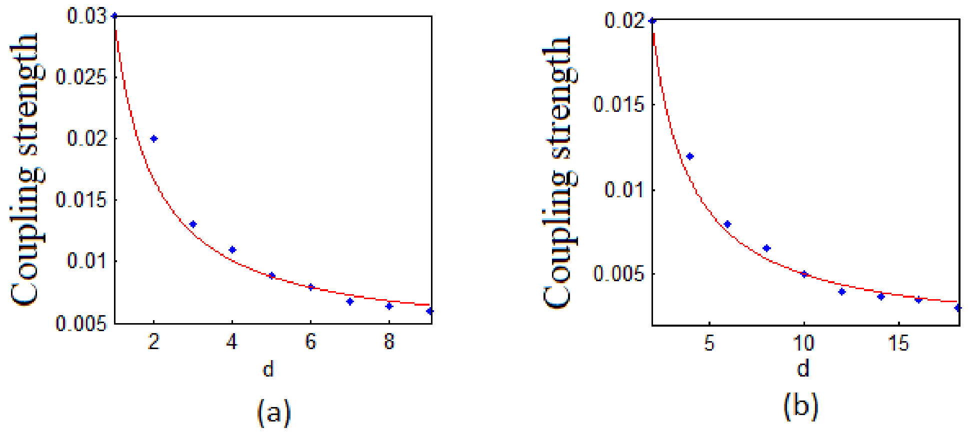

Figure 4. The evolution of the coupling strength with respect to the number of input links at each node in the regular network with unidirectional coupling (a), and bidirectional coupling (b)

4. Conclusion

The article gave results on the speed of synchronization between two regular network structures of reaction-diffusion systems of FitzHugh-Nagumo type. The results show that the coupling strength required for synchronization in both cases tends to decrease as the number of input links at each node in the network increases. That is, as the number of input links at each node increases, it becomes easier for synchronization to occur. In addition, the network with bidirectional coupling is more susceptible to synchronization than the network with unidirectional coupling on the same number of cells. In the next paper, the author will study the speed of synchronization of spiral solutions in the ring network with chemical connection.

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