KRONECKER PRODUCT AND ITS APPLICATION IN MATRIX OPERATIONS

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Abstract

Based on existing knowledge about Kronecker product of two matrices and matrix operations such as multiplying two matrices, computing the determinant of a square matrix and finding the inverse of a square matrix, the article clarifies the concept of Kronecker product of two matrices and some of its properties are related to matrix operations known above. The article will also present the application of Kronecker product in large-order matrix operations such as multiplying two matrices, computing the determinant of a square matrix and finding the inverse of a square matrix with specific illustrative examples. Applying Kronecker product in those matrix operations known above will significantly reduce the amount of calculation.

Keywords: application of Kronecker product, determinant of a square matrix, inverse of a matrix Kronecker product, matrix multiplication.

1. Introduction

1.1. Background and literature

The Kronecker product, named after the German mathematician Leopold Kronecker (1823 – 1891), is a special operator used in matrix algebra for multiplication of two matrices. This product is a concept having its origin in group theory and has wide applications in system theory (e.g., Shi and Yu, 2009; Shi, Fang, and Yan, 2009), system identification (e.g., Jódar and Abou-Kandil, 1989; Bahuguna, Ujlayan, and D. N. Pandey, 2007). The Kronecker product has also an important role in the linear matrix equation theory. The solution of the Sylvester and the Sylvester – like equations is a hotspot research area (e.g., Graham, 1981; Dehghan and Hajarian, 2008; Xie, Liu, and Yang, 2010).

In addition, sparse factorizations and Kronecker products are proving to be a very effective way to look at fast linear transforms. Researchers have taken the Kronecker methodology as developed for the fast Fourier transform and used it to build exciting alternatives. As computers get more powerful, researchers are more willing to entertain problems of high dimension and this leads to Kronecker products whenever low-dimension techniques are “tensored” together. Therefore, the Kronecker product decomposition problem/algorithm is currently receiving much attention from mathematicians. For example:

The necessary and sufficient condition that a non-singular 4×4 complex (or real) matrix can be written as a the Kronecker product of two 2×2 complex (or real) matrices were established by Shamar (Shamar, 1991).

A method of Kronecker product decomposition was introduced by Enríquez and O. Rosas-Ortiz in their paper (Enríquez and Rosas-Ortiz, 2013).

Yi Wu introduced a new method of Kronecker product decomposition in his paper. The simulation results in this paper show that the new method is far faster than the known method and the new method is very applicable for exact decomposition, fast decomposition, big matrix decomposition, and online decomposition of Kronecker products (Yi Wu, 2023).

1.2. The aim of the paper

When learning about Kronecker product, from the properties:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$|A \otimes B| = |A|^m |B|^n$$

I found that if a large-order matrix can be factored into the Kronecker product of two smaller-order matrices, then obviously the amount of computation required to multiply two large-order matrices or find the inverse of a large-order square matrix as well as computing the determinant of a large-order square matrix will be significantly reduced.

Since I position this article as reference for students, I will try to write clearly and simply so that even a first-year student can understand. And I will not go into the Kronecker product decomposition problem/algorithm but just focus on its application. Specifically, I will present the method of using Kronecker product to solve the problems of multiplying two large-order matrices, finding the inverse of a large-order square matrix or computing the determinant of a large-order square matrix with specific illustrative examples.

2. Mathematical preliminaries

2.1. Multiplication of two matrices

2.1.1. Definition

Given $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$. The matrix product $AB = [c_{ij}]$ is a matrix of order $m \times p$ such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \ (i = \overline{1,m}; j = \overline{1,p}).$$

2.1.2. Remark

Theoretically, to calculate an element c_{ij} we need n multiplications and $n - 1$ additions. The product AB has $m.p$ elements, so we need a total of $m.p.n$ multiplications and $m.p.(n - 1)$ additions.

2.1.3. Example Calculate AB with $A = \begin{bmatrix} 3 & 4 & 7 & 8 & 2 \\ 1 & 0 & 4 & 7 & 8 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 2 & 6 \\ 1 & 2 \\ 0 & 7 \\ 5 & 8 \end{bmatrix}$.

We have

$$AB = \begin{bmatrix} 3.4 + 4.2 + 7.1 + 8.0 + 2.5 & 3.5 + 4.6 + 7.2 + 8.7 + 2.8 \\ 1.4 + 0.2 + 4.1 + 7.0 + 8.5 & 1.5 + 0.6 + 4.2 + 7.7 + 8.8 \end{bmatrix} = \begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix}$$

To calculate an element of AB, we need 5 multiplications and 4 additions. The product AB has 4 elements, so we need a total of 20 multiplications and 16 additions.

2.2. Determinant of a square matrix

2.2.1. Definition

Let $A = [a_{ij}]$ is a square matrix of order n. The determinant of A, denoted by $|A|$, is determined by the formula:

$$|A| = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \text{ (Leibniz formula)}$$

where σ is a permutation of degree n and $sign(\sigma)$ is the sign of σ.

2.2.2. Calculate the determinant of a square matrix using Gaussian elimination

We know that for a large-order square matrix, to reduce the number of calculations, the determinant should be computed by Gaussian elimination instead of using Leibniz formula.

Let $A = [a_{ij}]$ is a square matrix of order n. We need nullify all the elements below the diagonal so that A becomes an upper triangular matrix.

At each below step (except the last step), if the element is at 1st row and 1st column equal 0 then we find the non-zero element in 1st column and then interchange those two rows. If no such element is found, the determinant is 0. Moreover, the number of multiplications and additions is the same in those steps.

Step 1: nullify a_{i1} $(i = \overline{2, n})$ in the first column. To nullify a_{i1}, we need n multiplications/additions. Cause there are $n - 1$ elements a_{i1} $(i = \overline{2, n})$ in the first column, we need $(n - 1)n$ multiplications/additions in this step.

Step 2: Repeat step 1 for square matrix of order $(n - 1)$ after deleting the first row and the first column of the result matrix in step 1. Thus we need $(n - 2)(n - 1)$ multiplications/additions in this step.

Continue the above process until we only have to nullify element in the first column and the last row of square matrix of order 2. We need 2 multiplications/additions in this step.

The last step: multiply the diagonal elements, we have $n - 1$ multiplications. Note that if the number of rows changed is odd, we must add a minus sign to the result of the determinant.

Theoretically, to A becomes an upper triangular matrix, we total need

$(n-1)n+(n-2)(n-1)\ +\cdots+1.2+n-1=\frac{(n-1)n(n+1)}{3}+n-1=\frac{n^3+2n-3}{3}$ multiplications and $\frac{(n-1)n(n+1)}{3}$ additions.

2.2.3. Example Compute $|A|$ with $A=\begin{bmatrix}8 & 1 & 5\\ 4 & 7 & 2\\ 1 & 3 & 9\end{bmatrix}$.

Applying Gaussian elimination, we get

$$\begin{bmatrix}8 & 1 & 5\\ 4 & 7 & 2\\ 1 & 3 & 9\end{bmatrix} \rightarrow \begin{bmatrix}8 & 1 & 5\\ 0 & 13/2 & -1/2\\ 0 & 23/8 & 67/8\end{bmatrix} \rightarrow \begin{bmatrix}8 & 1 & 5\\ 0 & 13/2 & -1/2\\ 0 & 0 & 447/52\end{bmatrix}$$

Thus $|A|=8.\left(\frac{13}{2}\right).\left(\frac{447}{52}\right)=447.$

We need 6 multiplications/additions in the first step and 2 multiplication/additions in the second step. Finally, we need 2 multiplications to multiply the diagonal elements. Therfore we need a total of 10 multiplications and 8 additions to compute $|A|$.

2.3. The inverse of a square matrix

2.3.1. Definition

Let $A=[a_{ij}]$ is a square matrix of order n. A is called invertible if there exists a square matrix B of order n such that $AB=BA=I_n$

Then the matrix B is uniquely determined by A, and is called the inverse of A, denoted by A^{-1}.

2.3.2. Find the inverse of a matrix using Gauss-Jordan elimination

Let $A=[a_{ij}]$ is a square matrix of order n. We use Gauss-Jordan elimination to tranform matrix $[A|I_n]$ of order $n\times 2n$ into matrix $[I_n|A^{-1}]$. It means we have to nullify for all elements $a_{ij}(i\neq j; i,j=\overline{1,n})$ and reduce $a_{ii}(i=\overline{1,n})$ to 1.

In step i below (except the last step), if the element is at ith row and ith column $(i=\overline{1,n})$ equal 0, we find the non-zero element in jth row and ith column $(j>i)$ and interchange those two rows. If no such element is found, the inverse of A is not exist. Moreover, the number of multiplications and additions is the same in those steps.

First, we nullify $a_{ij}\ (i\neq j; i,j=\overline{1,n})$. To nullify a_{i1}, we need $2n$ multiplications/additions. Cause there are n column and $(n-1)$ elements a_{ij} $(i\neq j; i,j=\overline{1,n})$ in every column, we need $2n^2(n-1)$ multiplications/additions in this step.

Finally, to reduce elements are at ist row and ist column $(i=\overline{1,n})$ to 1, we need $2n^2$ multiplications.

Theoretically, we total need $2n^2(n-1)+2n^2=2n^3$ multiplications and $2n^2(n-1)$ additions.

2.3.3. Example Find A^{-1} with $A=\begin{bmatrix}8 & 1 & 5\\ 4 & 7 & 2\\ 1 & 3 & 9\end{bmatrix}$.

Applying Gauss-Jordan elimination, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 8 & 1 & 5 & 1 & 0 & 0 \\ 4 & 7 & 2 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 8 & 1 & 5 & 1 & 0 & 0 \\ 0 & 13/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 23/8 & 67/8 & -1/8 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 8 & 0 & 66/13 & 14/13 & -2/13 & 0 \\ 0 & 13/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 447/52 & 5/52 & -23/52 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 152/149 & 16/149 & -88/149 \\ 0 & 13/2 & 0 & -221/447 & 871/894 & 26/447 \\ 0 & 0 & 447/52 & 5/52 & -23/52 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 19/149 & 2/149 & -11/149 \\ 0 & 1 & 0 & -34/149 & 67/447 & 4/447 \\ 0 & 0 & 1 & 5/447 & -23/447 & 52/447 \end{array}\right] = [I_3|A^{-1}]$$

Thus $A^{-1} = \begin{bmatrix} 19/149 & 2/149 & -11/149 \\ -34/149 & 67/447 & 4/447 \\ 5/447 & -23/447 & 52/447 \end{bmatrix}$.

We need 12 multiplications/additions in each step $1, 2, 3$. Finally, we need 18 multiplications to reduce elements are at ith row and ith column $(i = 1,2,3)$ to 1. Therefore we need a total of 54 multiplications and 36 additions.

2.4. Kronecker product of two matrices

2.4.1. Definition

Consider a matrix $A = [a_{ij}]$ of order $m \times n$ and a matrix $B = [b_{ij}]$ of order $r \times s$. The Kronecker product of the two matrices, denoted by $A \otimes B$, is defined as the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

$A \otimes B$ is seen to be a matrix of order $mr \times ns$. It has $m.n$ blocks, the $(i,j)th$ block is the matrix $a_{ij}B$ of order $r \times s$.

2.4.2. Remark

Theoretically, to calculate an element of $A \otimes B$ we only need one multiplication. The matrix $A \otimes B$ has $m.r.n.s$ elements, so we need a total of $m.r.n.s$ multiplications.

2.4.3. Example Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 4 & 8 \end{bmatrix}, B = \begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix}$. Then

$$A \otimes B = \begin{bmatrix} 3B & 2B & 4B \\ 6B & 4B & 8B \end{bmatrix} = \begin{bmatrix} 3\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} & 2\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} & 4\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} \\ 6\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} & 4\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} & 8\begin{bmatrix} 37 & 125 \\ 48 & 126 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 111 & 375 & 74 & 250 & 148 & 500 \\ 144 & 378 & 96 & 252 & 192 & 504 \\ 222 & 750 & 148 & 500 & 296 & 1000 \\ 288 & 756 & 192 & 504 & 384 & 1008 \end{bmatrix}$$

We only need a total of 24 multiplications to calculate $A \otimes B$.

2.4.4. Some properties (Graham, 1981)

i) $$(A \otimes B)(C \otimes D) = AC \otimes BD$$

where A, B, C, D are matrices of order $m \times n, r \times s, n \times p, s \times t$ respectively.

ii) $$|A \otimes B| = |A|^n |B|^m$$

where A, B are square matrices of order m and n respectively.

iii) $$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

where A, B are invertible matrices of order m and n respectively.

3. Application of Kronecker product in matrices operations

3.1. Application of Kronecker product in multiplication of two matrices

3.1.1. Theoretical basis

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$. If $A = A_1 \otimes A_2$, $B = B_1 \otimes B_2$ where A_i, B_i $(i = 1, 2)$ are respectively matrices of order $m_1 \times n_1, m_2 \times n_2, n_1 \times p_1, n_2 \times p_2$ such that

$$m_1.m_2 = m; n_1.n_2 = n; p_1.p_2 = p.$$

Then we have another way to calculate multiplication of AB:

$$AB = (A_1 \otimes A_2).(B_1 \otimes B_2) = (A_1B_1) \otimes (A_2B_2).$$

The representations $A = A_1 \otimes A_2$, $B = B_1 \otimes B_2$ may not be unique. However based on property 2.4.4.i above, it does not affect the result of the operation.

3.1.2. Example Calculate AB with

$$A = \begin{bmatrix} 3 & 4 & 7 & 8 & 2 \\ 1 & 0 & 4 & 7 & 8 \\ 6 & 8 & 14 & 16 & 4 \\ 2 & 0 & 8 & 14 & 16 \end{bmatrix}, B = \begin{bmatrix} 12 & 15 & 8 & 10 & 16 & 20 \\ 6 & 18 & 4 & 12 & 8 & 24 \\ 3 & 6 & 2 & 4 & 4 & 8 \\ 0 & 21 & 0 & 14 & 0 & 28 \\ 15 & 24 & 10 & 16 & 20 & 32 \end{bmatrix}$$

If we calculate the product AB directly, we need a total of 120 multiplications and 96 additions (see 2.1.2). Now we will compute AB using the Kronecker product.

We realize that

$$A=\begin{bmatrix}3&4&7&8&2\\1&0&4&7&8\\6&8&14&16&4\\2&0&8&14&16\end{bmatrix}=\begin{bmatrix}1\begin{bmatrix}3&4&7&8&2\\1&0&4&7&8\end{bmatrix}\\2\begin{bmatrix}3&4&7&8&2\\1&0&4&7&8\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}1\\2\end{bmatrix}\otimes\begin{bmatrix}3&4&7&8&2\\1&0&4&7&8\end{bmatrix}=A_1\otimes A_2.$$

And $$B=\begin{bmatrix}12&15&8&10&16&20\\6&18&4&12&8&24\\3&6&2&4&4&8\\0&21&0&14&0&28\\15&24&10&16&20&32\end{bmatrix}=\begin{bmatrix}3\begin{bmatrix}4&5\\2&6\\1&2\\0&7\\5&8\end{bmatrix}&2\begin{bmatrix}4&5\\2&6\\1&2\\0&7\\5&8\end{bmatrix}&4\begin{bmatrix}4&5\\2&6\\1&2\\0&7\\5&8\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}3&2&4\end{bmatrix}\otimes\begin{bmatrix}4&5\\2&6\\1&2\\0&7\\5&8\end{bmatrix}=B_1\otimes B_2.$$

Thus $AB=(A_1\otimes A_2).(B_1\otimes B_2)=(A_1B_1)\otimes(A_2B_2)$

$$=\left(\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3&2&4\end{bmatrix}\right)\otimes\left(\begin{bmatrix}3&4&7&8&2\\1&0&4&7&8\end{bmatrix}\begin{bmatrix}4&5\\2&6\\1&2\\0&7\\5&8\end{bmatrix}\right)$$

$$=\begin{bmatrix}3&2&4\\6&4&8\end{bmatrix}\otimes\begin{bmatrix}37&125\\48&126\end{bmatrix}=\begin{bmatrix}111&375&74&250&148&500\\144&378&96&252&192&504\\222&750&148&500&296&1000\\288&756&192&504&384&1008\end{bmatrix}.$$

We need 6 multiplications to calculate A_1B_1 and 20 multiplications, 16 additions to calculate A_2B_2 (see 2.1.2). Finally, we need 24 multiplications to calculate $(A_1B_1)\otimes(A_2B_2)$ (see 2.4.2). Thus we need 50 multiplications and 16 additions to calculate AB by using the Kronecker product.

With the above example, we can see that if we calculate AB using the Kronecker product, the number of multiplications and additions required is significantly reduced.

3.2. Application of Kronecker product in calculating determinant of square matrices

3.2.1. Theoretical basis

Let C is a square matrix of order n. If $C=A\otimes B$ where A,B are respectively square matrices of order $m,\ p$ such that $m.p=n$.

Then we have another way to calculate determinant of A:

$$|C|=|A\otimes B|=|A|^p|B|^m.$$

The representation $C=A\otimes B$ may not be unique. However based on property 2.4.4.ii above, it does not affect the result of the operation.

3.2.2. Example Compute $|C|$ with $C = \begin{bmatrix} 8 & 1 & 5 & 24 & 3 & 15 \\ 4 & 7 & 2 & 12 & 21 & 6 \\ 1 & 3 & 9 & 3 & 9 & 27 \\ 40 & 5 & 25 & 16 & 2 & 10 \\ 20 & 35 & 10 & 8 & 14 & 4 \\ 5 & 15 & 45 & 2 & 6 & 18 \end{bmatrix}$.

If we compute $|C|$ using Gaussian elimination, we need a total of 75 multiplications and 70 additions (see 2.2.2). Now we will compute $|C|$ using the Kronecker product.

We realize that $C = \begin{bmatrix} 1\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} & 3\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} \\ 5\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} & 2\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \otimes \begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} = A \otimes B.$$

Thus $|C| = |A \otimes B| = |A|^3|B|^2 = \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix}^3 \cdot \begin{vmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{vmatrix}^2 = -438980373.$

We need 3 multiplications, 2 additions to calculate $|A|$ and 10 multiplications, 8 additions to calculate $|B|$ (see 2.2.2). Finally, we need 4 multiplications to calculate $|A|^3|B|^2$. So we need 17 multiplications and 10 additions to calculate AB by using the Kronecker product.

With the above example, we can see that if we calculate $|C|$ using the Kronecker product, the number of multiplications and additions required is significantly reduced.

3.3. Application of Kronecker product in finding the inverse of a square matrix

3.3.1. Theoretical basis

Let C be an invertible matrix of order n. If $C = A \otimes B$ where A, B are respectively invertible matrices of order m, p such that $m.p = n$.

Then we have another way to calculate the inverse of C:

$$C^{-1} = (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

The representation $C = A \otimes B$ may not be unique. However based on property 2.4.4.iii above, it does not affect the result of the operation.

3.3.2. Example Find C^{-1} with $C = \begin{bmatrix} 8 & 1 & 5 & 24 & 3 & 15 \\ 4 & 7 & 2 & 12 & 21 & 6 \\ 1 & 3 & 9 & 3 & 9 & 27 \\ 40 & 5 & 25 & 16 & 2 & 10 \\ 20 & 35 & 10 & 8 & 14 & 4 \\ 5 & 15 & 45 & 2 & 6 & 18 \end{bmatrix}$.

If we calculate C^{-1} using Gauss-Jordan elimination, we need a total of 432 multiplications and 360 additions (see 2.3.2). Now we will compute C^{-1} using the Kronecker product.

We realize that $C = \begin{bmatrix} 1\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} & 3\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} \\ 5\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} & 2\begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \otimes \begin{bmatrix} 8 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 9 \end{bmatrix} = A \otimes B.$$

Applying Gauss-Jordan elimination to calculate A^{-1}, B^{-1} we get

$$A^{-1} = \begin{bmatrix} \frac{-2}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{-1}{13} \end{bmatrix}, B^{-1} = \begin{bmatrix} \frac{19}{149} & \frac{2}{149} & \frac{-11}{149} \\ \frac{-34}{447} & \frac{67}{447} & \frac{4}{447} \\ \frac{5}{447} & \frac{-23}{447} & \frac{52}{447} \end{bmatrix}.$$

Thus

$$C^{-1} = (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} = \begin{bmatrix} \frac{-2}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{-1}{13} \end{bmatrix} \otimes \begin{bmatrix} \frac{19}{149} & \frac{2}{149} & \frac{-11}{149} \\ \frac{-34}{447} & \frac{67}{447} & \frac{4}{447} \\ \frac{5}{447} & \frac{-23}{447} & \frac{52}{447} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-38}{1937} & \frac{-4}{1937} & \frac{22}{1937} & \frac{57}{1937} & \frac{6}{1937} & \frac{-33}{1937} \\ \frac{68}{5811} & \frac{-134}{5811} & \frac{-8}{5811} & \frac{-34}{1937} & \frac{67}{1937} & \frac{4}{1937} \\ \frac{-10}{5811} & \frac{46}{5811} & \frac{-8}{447} & \frac{5}{1937} & \frac{-23}{1937} & \frac{4}{149} \\ \frac{95}{1937} & \frac{10}{1937} & \frac{-55}{1937} & \frac{-19}{1937} & \frac{-2}{1937} & \frac{11}{1937} \\ \frac{-170}{5811} & \frac{335}{5811} & \frac{20}{5811} & \frac{34}{5811} & \frac{-67}{5811} & \frac{-4}{5811} \\ \frac{25}{5811} & \frac{-115}{5811} & \frac{20}{447} & \frac{-5}{5811} & \frac{23}{5811} & \frac{-4}{447} \end{bmatrix}$$

We need 16 multiplications, 8 additions to calculate A^{-1} and 54 multiplications, 36 additions to calculate B^{-1} (see 2.3.2). Finally, we need 36 multiplications to calculate $A^{-1} \otimes B^{-1}$ (see 2.4.2). Thus, we need 106 multiplications and 44 additions to calculate C^{-1} by using the Kronecker product.

With the above example, we can see that if we calculate C^{-1} using the Kronecker product, the number of multiplications and additions required is significantly reduced.

4. Conclusion

The article has clarified the concept of Kronecker product of two matrices and some useful properties. The article has also clarified how to use Kronecker product to calculate multiplication of two matrices, calculate determinant and find inverse matrix of a square matrix.

Through the article we can see two advantages when using Kronecker product to solve those problems. First, the Kronecker product is simpler to implement than the usual multiplication of two matrices. Second, instead of performing the calculation directly on large matrices, if we use Kronecker, we can transfer the calculation to smaller matrices. These advantages will help us significantly reduce the number of calculations. However, the article also has limitation that the above applications can only be applied to matrices that can be analyzed as Kronecker products of matrices of smaller order. In addition, the article also does not prove the superiority of using Kronecker product in finding the multiplication of two matrices, determinant and inverse matrix of a square matrix in general case, but only illustrates with specific examples. These problems will hopefully be solved in my next article.

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