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Continuous dependence on parameters of second order differential inclusion and self-adjoint operator

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ABSTRACT

In this paper, we establish compactness and continuous dependence on parameters for solution-set of the second order differential inclusion including self-adjoint operator in the form

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t),\mu), & (t,x) \in [0,T) \times \Omega \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, & x \in \Omega, \end{cases}$$

where $\mathcal{A}$ is a self-adjoint operator. We use the spectral theory on Hilbert spaces to obtain formulation for mild solutions. Using the mild solution formula together with a measure of noncompactness with values in an ordered space, we construct useful bounds for solution operators. Then, we establish necessarily upper semi-continuous and condensing settings, which mainly help to obtain the global existence of mild solutions and the compactness of the mild solution set. Finally, we provide a brief discussion on the continuous dependence of the solution-set on parameter μ.

Keywords: *multi-function, measure of compactness, differential inclusion, Self-Adjoint operator*

1 Introduction

Let Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega$ in Euclidean space $\mathbb{R}^N$ and T be a positive number. We first consider the following initial value problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t)), & (t,x) \in [0,T) \times \Omega, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial t^2}$ denote the symbols for the first and second-order derivatives with respect to the variable t, respectively; $\mathcal{A}$ is a self-adjoint operator on the Hilbert space $\mathbb{H}$, namely, $\langle \mathcal{A}^j u, w\rangle = \langle u, \mathcal{A}^j w\rangle$ for all $j \in \{1,2\}$; and F is a multi-valued mapping which is called source function.

Differential equations and inclusions have recently received a lot of interest due to their numerous applications in economics, control theory, physics, and other fields (see e.g. [3, 7, 11, 18, 19]). There have been numerous studies on the existence and the stability of the solution of the problem with the source single-valued function or with non-integer order derivatives, in the literature [1, 2, 5–17].

In 2016, Anh et al. [1], studied the following fractional differential equation with a multi-valued source function

$$\partial_t^\alpha x(t) - Ax(t) \in F(t,x,x_t), \quad t \in (0,\infty),\ \alpha \in (0,1), \tag{1.2}$$

involving impulsive effects. They demonstrated the global solvability and weakly asymptotic stability of solutions by examining their behavior on the half-line. This equation was also studied in [5]. Phong and Lan (see [12]) are interested in the retarded fractional evolution equation

$$\partial_t^\alpha u(t) - Au(t) \in F(t,u_t), \quad t \in (0,\infty), \alpha \in (0,1), \tag{1.3}$$

under the condition

$$u(s) = \varphi(s), \quad s \in [-h,0], h > 0,$$

where A is a closed linear mapping in a Banach space E, F is a multi-valued mapping and φ is the history of the solutions. When F super-linear, they proved the existence of decay global-solution. However, a recurrent issue in control theory is that F is a multi-valued mapping. Aside from the presence and continuity of the solution set, the compactness of the solution set is frequently of relevance. When the input data F is noisy due to a parameter, we must evaluate the solution set's continuous dependency on the parameter.

In [10], Ngoc and Tri investigated the existence and compactness of the solution-set to the following fractional pseudo-parabolic equation:

$$\begin{cases} \partial_t^\alpha w + \kappa(-\Delta)^{\gamma_1}\partial_t^\alpha w + (-\Delta)^{\gamma_2} w & \in & F(t,u), & t\in(0,T),\ x\in\Omega, \\ w(t,x) & = & 0, & t\in(0,T),\ x\in\partial\Omega, \\ w(0,x) & = & \varphi(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where ∂_t^α denotes the Caputo derivative of fractional order $\alpha\in(0,1)$ over time. Using asymptotic behaviors of the Mittag-Leffler functions, the authors constructed useful bounds for the solution-set to prove the compactness and continuous dependence on parameters of solutions-set of equation (1.4).

In [16], Tuan provided a regularized problem for bi-parabolic equation when the observed data are obtained in L^p ($p\neq 2$)

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = F(x,t,u(x,t)), \text{ in } \Omega\times(0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_t = \frac{\partial u}{\partial t}$ and F is a single-valued function, in particular, $F(t,x) = \varphi(t)f(x)$ and Tuan introduced the error between the Fourier regularized solution and the exact solution in L^p spaces.

The goal of this research is to investigate the initial value problem for differential inclusions (1.1). We demonstrate the existence and compactness of the solution set and describe how the solutions to the following parameterized problems rely on the parameter μ in a metric space (E,d). It is more obvious that we consider the following equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t,x),\mu), \ (t,x)\in(0,T]\times\Omega \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, \qquad x\in\Omega, \end{cases} \tag{1.6}$$

In addition to widely utilized approaches such as Fourier expansion of an element in Hilbert space evaluations, Gronwall's inequality. we use a measure of noncompactness β in the ordered space generated by a convex cone to consider the existence of fixed points of the β-condensing multi-map. To the extent of our knowledge, There are few works on differential inclusions using self-adjoint operators of fractional order and methodologies based on the noncompactness measure that takes values in cones.

Let $\mathbb{H}$ be a Hilbert space, we denote by $KV(\mathbb{H})$ (resp., $b(\mathbb{H})$) the all convex and compact (resp., bounded) subsets of $\mathbb{H}$ and consider problem (1.1) with the multi-valued function $F : [0,T] \times \mathbb{H} \to KV(\mathbb{H})$ under the following conditions (H):

(Ha) for $v \in \mathbb{H}$, there is a measurable function $f_v(.) : [0,T] \to \mathbb{H}$ satisfying $f_v(t) \in F(t,v)$;

(Hb) $F(t,.) : \mathbb{H} \to KV(\mathbb{H})$ is upper semicontinuous (usc, brief) for a.e. $t \in [0,T]$;

(Hc) there exists a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u)||| := \sup_{v \in F(t,u)} \|v\|_{\mathbb{H}} \leq \alpha(t)(1 + \|u\|_{\mathbb{H}}) \text{ for a.e. } t \in (0,T) \text{ and } \forall u \in \mathbb{H};$$

(Hd) there is $B \in L^1((0,T);\mathbb{R})$ satisfying

$$\beta(F(t,D)) \leq B(t)\beta(D) \text{ for a.e. } t \in (0,T) \text{ for all } D \in b(\mathbb{H}),$$

where β is noncompactness measure in $\mathbb{H}$ that is defined by $\beta(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}$.

Our work will be displayed as follows. In the following part, we will review some fundamental characteristics of multi-valued analysis. Section 3 discusses the global existence of mild solutions as well as the compactness of the solution set for the problem (1.1). Finally, we talk about the continuous dependence parameters μ of the solution set (1.6).

2 Preliminaries

We begin by describing some of the notations used throughout this paper. Let $\dot{\mathbb{N}} = \mathbb{N}\backslash\{0\}$ and $\mathcal{P}(X)$ ($b(X), k(X)$, resp.) be the all nonempty (bounded, compact, resp.) subsets of X. Let $\mathbb{H}$ be a separable Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_{\mathbb{H}} = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. The space of all continuous maps from $[0,T]$ into $\mathbb{H}$ is denoted by $\mathcal{C}([0,T];\mathbb{H})$. The norm in $\mathcal{C}([0,T],\mathbb{H})$ is defined by

$$\|u\|_{\mathcal{C}([0,T],\mathbb{H})} = \sup_{t \in [0,T]} \|u(t,.)\|_{\mathbb{H}}.$$

Let (X,ρ) be a metric space and G be a subset of X. We denote the distance between a point $x \in X$ and G by $\text{dist}(x,G) := \inf\{\rho(x,y) : y \in G\}$, and the ε-neighbourhood of G by $\mathcal{L}_{\varepsilon,\rho}(G) := \{y \in X : \text{dist}(y,G) < \varepsilon\}$ (in short, $\mathcal{L}_\varepsilon(G)$).

To establish our primary findings, we require certain basic multi-valued analytical features, which may be found in [4]. Let us review the concepts and attributes that will be used in the subsequent sections.

Definition 2.1. [4, Definition 2.1.1] Let E be a Banach space and $(C, \preceq)$ a partially ordered set. A map $\phi : Y \subset \mathcal{P}(E) \to C$ is said to be a *measure of noncompactness* (MNC) in Y if $\phi(\overline{co}(D)) = \phi(D)$ for all $D \in \mathcal{Y}$. A multi-mapping $F : E \to \mathcal{Y}$ is called *condensing* to ϕ (in short, ϕ-condensing) if $D \in Y$ with $\phi(D) \preceq \phi(F(D))$ then D is relatively compact in E.

Let G be a subset of a metric space (E, d) and ϵ be a positive number. A subset A of E is said to be ϵ-net of G if $G \subset \bigcup_{x\in A} \{y \in E : d(x, y) < \epsilon\}$. If A is finite, A is called a *finite ϵ-net*. We need the Hausdorff measure β which defined in [4, Definition 2.1.1], i.e., $\beta(G) = \inf\{\epsilon > 0 : G \text{ has a finite } \epsilon\text{-net}\}$.

Lemma 2.2. [4, Definition 2.1.1] *Let E be a Banach space and β a Hausdorff MNC defined on family $\mathcal{F}$ of subsets of E. Then β has the following properties:*

(a) monotone: if $D_1 \subset D_2$ implies $\beta(D_1) \leq \beta(D_2)$, for all $D_1, D_2 \in \mathcal{F}$.

(b) algebraically semiadditive: if $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$ for all $D_1, D_2 \in \mathcal{F}$.

(c) nonsingular: if $\beta(\{a\} \cup D) = \beta(D)$ for all $a \in E, D \in \mathcal{F}$.

(d) regular: $\beta(D) = 0$ if and only if D is relatively compact, $D \in \mathcal{F}$.

(e) semi-homogeneity: that is $\beta(\lambda D) = |\lambda|\beta(D)$ for all $\lambda \in \mathbb{R}, D \in \mathcal{F}$.

Definition 2.3. [4, Corollary 1.1.1] Let X and Y be topological spaces. A multi-valued mapping $F : X \to \mathcal{P}(Y)$ is upper semicontinuous at the point $u \in X$ if, for every open set $W \subset Y$ such that $F(x) \subset W$, there exists a neighborhood $V(x)$ of u with property that $F(V(u)) \subset W$. A multi-value mapping is called *upper semicontinuous* (usc) if it is upper continuous at every point $u \in X$.

Let $(E, d), (F, \rho)$ be metric spaces, it is clear that a multi-valued mapping f form a metric space (E, d) into (F, ρ) is usc at point $x \in E$ iff for any $\epsilon > 0$, there exists $\delta > 0$ such that $f(\nu) \subset \mathcal{L}_{\epsilon,\rho}(f(x))$ for all $\nu \in \mathcal{L}_{\delta,d}(x)$.

For multi-valued mapping $\Phi : E \to \mathcal{P}(E)$, we denote by $\mathrm{Fix}(\Phi)$ the set of the all fixed points of $\mathcal{M}$, i.e., $\mathrm{Fix}(\Phi) = \{x \in E : x \in \Phi(x)\}$.

Lemma 2.4. [4, Corollary 3.3.1] *If D is a closed convex subset of Banach space E and $\Phi : D \to KV(D)$ is a closed φ-condensing multi-valued mapping, where φ is a nonsingular MNC defined on subsets of D, then* $\mathrm{Fix}(\Phi)$ *is nonempty.*

Lemma 2.5. [4, Propositions 3.5.1] *Let D be a closed subset of a Banach space E and $\Phi : D \to k(D)$ a closed multi-valued mapping, which is ϕ-condensing on every bounded subset of D, where ϕ is a monotone MNC. If* $\mathrm{Fix}(\Phi)$ *is bounded then it is compact.*

Lemma 2.6. [4, Propositions 3.5.2] *Let X be a closed subset of a Banach space E, β be a monotone MNC in E, Y be a metric space, and $G : Y \times X \to k(E)$ be a closed multi-valued mapping which is β-condensing in the second variable and such that $F(\lambda) := \mathrm{Fix}\, G(\lambda, \cdot) \neq \emptyset$, for all $\lambda \in Y$. Then the multi-valued mapping $F : Y \to \mathcal{P}(E)$ is usc.*

Definition 2.7. ([4, Definition 4.2.1]) A sequence $\{x_n\}_{n\in\mathbb{N}} \subset L^1([0, d], E)$ (E is a Banach space) is called

1. *integrably bounded* if there is $q \in L^1([0, d], \mathbb{R})$ such that

$$\|x_n(t)\|_E \leq q(t) \text{ for a.e } t \in [0, d] \; \forall n \in \mathbb{N};$$

2. *semicompact* if it is integrably bounded and the set $\{x_n(t)\}_{n\in\mathbb{N}}$ is relatively compact for a.e $t \in [0, d]$.

In addition to the above mentioned basic properties of multi-valued analysis, we also use the Gronwall's inequality presented in the following lemma.

Lemma 2.8. (Gronwall) *Let $a \in (0, \infty)$, $0 < T \leq \infty$, and continuous functions $\alpha, \beta : [0, T] \to \mathbb{R}_+$ satisfying $\int_0^T \beta(s)ds < \infty$, and $\sup_{t\in[0,T]} \beta(t) < \infty$, $0 \leq \gamma \leq \xi \leq T$, and*

$$\beta(t) \leq a + \int_t^T \alpha(s)\beta(s)ds \;\left(\textit{resp.}, \; \beta(t) \leq a + \int_0^t \alpha(s)\mu(s)ds\right), \quad t \in [0, T].$$

Then $\beta(t) \leq a e^{\int_t^{\xi} \alpha(s)ds} \left(\textit{resp.}, \; \beta(t) \leq a e^{\int_{\gamma}^{t} \alpha(s)ds}\right)$ *for all* $t \in [0, T]$.

3 Main results

In the first part, we introduce the mild solution for problem (1.1). In the next part, we prove the existence and compactness of the solution-set. In the final part, we explore the parameter's continuous dependence on the solution-set of the problem (1.6).

3.1 Mild solution

For every $u \in \mathcal{C}([0,T];\mathbb{H})$, we denote

$$\mathscr{S}_F(u) = \left\{ f \in L^1((0,T);\mathbb{H}) | f(t,.) \in F(t,u), \text{ for a.e. } t \in (0,T) \right\}. \tag{3.1}$$

It is clear that $u = u(t,.)$ is a solution of Problem (1.1) if and only if there exists $f \in \mathscr{S}_F(u)$ satisfying

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) + 2\mathcal{A}\frac{\partial}{\partial t} u(t,x) + \mathcal{A}^2 u(t,x) = f(t,x), & (t,x) \in (0,T] \times \Omega \\ u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0, & x \in \Omega. \end{cases} \tag{3.2}$$

Assume that $\phi_\lambda \in \mathbb{H}$ is the eigen-function corresponding to the eigenvalue λ of the operator $\mathcal{A}$. Taking the inner product of both sides of (3.2) with ϕ_λ we obtain that

$$\frac{d^2}{dt^2}\langle u(t), \phi_\lambda\rangle + 2\lambda \frac{d}{dt}\langle u(t), \phi_\lambda\rangle + \lambda^2 \langle u(t), \phi_\lambda\rangle = \langle f(t), \phi_\lambda\rangle. \tag{3.3}$$

By the method of constant variation, from (3.3) it follows that

$$\langle u(t), \phi_\lambda\rangle = \int_0^t (t-s)\langle f(s), \phi_\lambda\rangle e^{-\lambda(t-s)} ds. \tag{3.4}$$

Throughout this paper, let ϕ_n, $n \in \dot{\mathbb{N}}$, be the eigenfunction corresponding to the eigenvalues λ_n satisfying $0 < \lambda_1 < \lambda_2 < \ldots$, and $\lim_{n\to\infty} \lambda_n = \infty$. Furthermore, assume that $\{\phi_n\}_{n \in \dot{\mathbb{N}}}$ is an orthonormal basis of $\mathbb{H}$. If Problem (3.2) has a solution $u \in \mathcal{C}([0,T],\mathbb{H})$, then

$$u(t) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle f(s), \phi_n\rangle \phi_n e^{-\lambda_n(t-s)} ds. \tag{3.5}$$

This suggests to define the mild solution of the problem (1.1) as follows:

Definition 3.1. A mapping $x \in \mathcal{C}([0,T];\mathbb{H})$ is called a mild solution of Problem (1.1) if following conditions hold

(i) $x(0,\cdot) = \frac{\partial}{\partial t} x(0,\cdot) = 0$, and

(ii) there is $f \in \mathscr{S}_F(u)$ such that

$$x(t,.) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle f(s), \phi_n\rangle \phi_n(.) e^{-\lambda_n(t-s)} ds \text{ for all } t \in [0,T]. \tag{3.6}$$

Since $f \in L^1((0,T);\mathbb{H})$, it is clear that (3.6) is well defined and $x(t,.) \in \mathbb{H}$ for a.e $t \in [0,T]$.

3.2 Upper semicontinuous and condensing settings

For $g \in L^1((0,T);\mathbb{H})$ we define

$$\Phi(g)(t,.) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle g(s), \phi_n\rangle \phi_n(.) e^{-\lambda_n (t-s)} ds. \tag{3.7}$$

It is clear that Φ is well defined. In this subsection, we establish the usc and β-condensing properties of the multi-valued mapping $\Phi \circ \mathscr{S}_F$.

Lemma 3.2. *Let $\{f_n\} \subset L^1((0,T);\mathbb{H})$ be a semicompact sequence. Then,*

a) the set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is equicontinuous.

b) the set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{H})$ and $\Phi(f_n) \to \Phi(f_0)$ if $\{f_n\}$ weakly converges to f_0.

Proof. We first begin with proving the assertion a). Assume that $t, t' \in [0,T]$ satisfying $0 \le t < t' \le T$. We write

$$\Phi(f_n)(t) - \Phi(f_n)(t') = \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t) - \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t'), \tag{3.8}$$

here

$$\mathcal{R}_j(n)(t) = \int_0^t \alpha_n(t,s,j) ds, \quad \alpha_n(t,s,j) = (t-s)\langle f_n(s), \phi_j\rangle \phi_j e^{-\lambda_j (t-s)}.$$

Then, we get

$$\mathcal{R}_j(n)(t) - \mathcal{R}_j(n)(t') = \int_0^t (\alpha_n(t,s,j) - \alpha_n(t',s,j))\, ds - \int_t^{t'} \alpha_n(t',s,j) ds. \tag{3.9}$$

Using the mean value theorem for function $t \mapsto (t-s)e^{-\lambda_j(t-s)}$, we obtain

$$(t-s)e^{-\lambda_j(t-s)} - (t'-s)e^{-\lambda_j(t'-s)} = (1-\lambda_j(\xi_j - s))e^{-\lambda_j(\xi_j - s)}(t-t') \text{ for some } \xi_j \in (t,t').$$

Therefore, from the condition $0 \le s \le t < \xi_j \le t' \le T$ it implies that the set $\{\mu_j : j = 1,2,...\}$, here $\mu_j = (1-\lambda_j(\xi_j - s))^2 e^{-2\lambda(\xi_j - s)}$, is bounded. Hence,

$$\begin{aligned}\left\| \sum_{j=1}^{\infty} (\alpha_n(t,s,j) - \alpha_n(t',s,j)) \right\|_{\mathbb{H}}^2 &= \sum_{j=1}^{\infty} (1-\lambda_j(\xi_j - s))^2 \langle f_n(s), \phi_j\rangle^2 e^{-2\lambda_j(\xi_j - s)} |t'-t|^2 \\ &\le C_1 \sum_{j=1}^{\infty} \langle f_n(s), \phi_j\rangle^2 |t'-t|^2 \\ &= C_1 \|f_n(s)\|_{\mathbb{H}}^2 |t'-t|^2. \end{aligned} \tag{3.10}$$

Further, we have

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}\alpha_n(t',s,j)\right\|_{\mathbb{H}}^2 &= \sum_{j=1}^{\infty}(t-s)^2\langle f_n(s),\phi_j\rangle^2 \\ &\le C_2\sum_{j=1}^{\infty}\langle f_n(s),\phi_j\rangle^2 \\ &= C_2\|f_n(s)\|_{\mathbb{H}}^2. \end{aligned} \tag{3.11}$$

Combination of (3.11), (3.10), (3.9) and (3.8) it show that

$$\|\Phi(f_n)(t)-\Phi(f_n)(t')\|_{\mathbb{H}} \le \sqrt{C_2}\int_t^{t'}\|f_n(s)\|_{\mathbb{H}}ds + \sqrt{C_1}\int_0^t\|f_n(s)\|_{\mathbb{H}}ds|t'-t| \tag{3.12}$$

Since the sequence $\{f_n\}$ is integrably bounded, there exists $\alpha \in L^1([0,T],\mathbb{R})$ such that $\|f_n(s)\|_{\mathbb{H}} \le \alpha(s)$ for a.e $s \in [0,T]$ and for all $n \in \dot{\mathbb{N}}$. From (3.12) we evaluate

$$\|\Phi(f_n)(t)-\Phi(f_n)(t')\|_{\mathbb{H}} \le C|t'-t| \quad \text{for all } n=1,2,... \tag{3.13}$$

This deduce the assertion a).

b) e will prove the set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is bounded at any point $t \in [0,T]$. Indeed, for every $t \in [0,T]$, since $\{f_n\}$ is integrally bounded we get

$$\begin{aligned}\|\Phi(f_n)(t)\|_{\mathbb{H}} &\le C_0\int_0^T\|f_n(s)\|_{\mathbb{H}}ds \\ &\le C_0\int_0^T\alpha(s)ds = C \ \ \forall n \in \dot{\mathbb{N}}. \end{aligned} \tag{3.14}$$

By Arzela-Ascoli theorem, it implies that $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relative compact in $\mathcal{C}([0,T],\mathbb{H})$. The second assertion b) is a consequence of the first assertion with the note that Φ is bounded linear mapping from $L^1((0,T);\mathbb{H})$ to $\mathcal{C}([0,T];\mathbb{H})$. □

Using the upper semicontinuous assumption (Hb) of F and applying Mazur's theorem, we obtain the following lemma.

Lemma 3.3. *Let $\{v_n\}_{n\in\dot{\mathbb{N}}} \subset \mathcal{C}([0,T];\mathbb{H})$ and $\{f_n\}_{\in\dot{\mathbb{N}}} \subset L^1((0,T);\mathbb{H})$ satisfying $f_n \in \mathscr{S}_F(v_n)$ for all $n \ge 1$. Then, if $v_n \to v$ and $\{f_n\}$ weakly converges to f, $f \in \mathscr{S}_F(v)$.*

The closed property of the multioperator $\Phi \circ \mathscr{S}_F$ which consequence of the use Lemma 3.2 and Lemma 3.3.

Lemma 3.4. *Assume that the condition (H) is satisfied. Then* $\Phi \circ \mathscr{S}_F$ *is closed multioperators from* $L^1((0,T);\mathbb{H})$ *into* $\mathcal{C}([0,T],\mathbb{H})$.

Proof. We show the closed property of $\Phi \circ \mathscr{S}_F$. Assume that sequences $\{v_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ in $\mathcal{C}([0,T];\mathbb{H})$ satisfying

$$\lim_{n\to\infty} v_n = v, \quad z_n \in \Phi \circ \mathscr{S}_F(v_n) \text{ and } \lim_{n\to\infty} z_n = z.$$

We will prove that $z \in \Phi \circ \mathscr{S}_F(v)$. Indeed, let $\{f_n\}$ be an arbitrary sequence in $L^1((0,T);\mathbb{H})$ satisfying $f_n \in \mathscr{S}_F(v_n)$ and $z_n = \Phi(f_n)$. From the condition (Hc) it follows that $\{f_n\}$ is integrally bounded. Further, from the condition (Hd) it follows that $\{f_n\}$ is semicompact and also weakly compact in $L^1((0,T);\mathbb{H})$ (see [4, Theorem 5.1.2]). Without loss of generality, we may assume that $\{f_n\}$ weakly converges to $f \in L^1((0,T);\mathbb{H})$. Using Lemma 3.2, we get $\Phi(f_n) \to \Phi(f) = z$, so by Lemma 3.3 we deduce $z \in \Phi \circ \mathscr{S}_F(v)$. □

The following lemma is a consequence of Lemma 3.2 and Lemma 3.4.

Lemma 3.5. *Assume the condition (H). Then, the multioperator* $\Phi \circ \mathscr{S}_F$ *is usc.*

Next, we present the condensing property of the multioperator $\Phi \circ \mathscr{S}_F$ associated with a suitable measure of noncompactness. Let $\mathcal{D} \subset \mathcal{C}([0,T],\mathbb{H})$, we denote by $\Delta(\mathcal{D})$ the family of all denumerable subsets of $\mathcal{D}$. Let L be a positive constant, we define

$$\nu_L(\mathcal{D}) = \max_{\mathcal{Q}\in\Delta(\mathcal{D})} \big(\gamma_L(\mathcal{Q}); \operatorname{mod}_{\mathrm{C}}(\mathcal{Q})\big),$$

where

$$\gamma_L(\mathcal{Q}) = \sup_{t\in[0,T]} e^{Lt}\beta(\mathcal{Q}(t)), \quad \operatorname{mod}_{\mathrm{C}}(\mathcal{Q}) = \lim_{\delta\to 0}\sup_{v\in D}\max_{|t'-t|\leq\delta} \|v(t') - v(t)\|,$$

$\mathcal{Q}(t) = \{w(t) : w \in \mathcal{Q}\}$. The MNC ν_L has the all properties which presented in Lemma 2.2. The reader can find their proofs in [4, Example 2.1.4].

Lemma 3.6. *Assume (H),* $\mathscr{S}_F : \mathcal{C}([0,T];\mathbb{H}) \to \mathcal{P}(L^1(0,T);\mathbb{H})$ *defined by* (3.1) *and* Φ *given by* (3.7). *Then, we can find* $L > 0$ *such that* $\Phi \circ \mathscr{S}_F$ *is* ν_L*-condensing.*

Proof. Let D be a bounded subset of $\mathcal{C}([0,T];\mathbb{H})$ satisfying

$$\nu_L(D) \preceq \nu_L(\Phi \circ \mathscr{S}_F), \tag{3.15}$$

here the order $\preceq$ is taken in $\mathbb{R}^2$ induced by the positive cone $\mathbb{R}_+ \times \mathbb{R}_+$. We will prove that D is relatively compact. Let $\{v_n\}$ be any sequence in D, we set $g_n(t,.) = \Phi(f_n)(t,.)$ with $f_n \in \mathscr{S}_F(v_n)$ and

$$\nu_L(\{g_n : n \geq 1\}) = (\gamma_L(\{g_n : n \geq 1\}); \mathrm{mod}_C(\{g_n : n \geq 1\}))\,,$$

the number L will be determined later. We have

$$\begin{aligned}
&e^{-Lt}\beta(\{g_n(t,.) : n \geq 1\}) \\
&= e^{-Lt}\beta\left(\left\{\sum_{j=1}^{\infty}\left(\int_0^t (t-s)\langle f_n(s),\phi_j\rangle e^{-\lambda(t-s)}ds\right)\phi_j(.) : n \geq 1\right\}\right) \\
&\leq C_0 e^{-Lt}\int_0^t \beta\left(\{f_n(s) : n \geq 1\}\right) ds \\
&\leq C_1 \sup_{s\in[0,T]}\left(e^{-Ls}\beta(\{v_n(s,.) : n \geq 1\})\right)\int_0^t s^{\gamma_1}e^{-L(t-s)}ds, \qquad (3.16)
\end{aligned}$$

where we have used β-regularity condition (Hd) in the last estimate. From the above inequality we obtain

$$\gamma_L(\{g_n : n \geq 1\}) \leq C_1\left(\sup_{\xi\in[0,T]}\int_0^{\xi}\tau^{\gamma_1}e^{-L(\xi-\tau)}d\tau\right)\gamma_L(\{v_n : n \geq 1\}). \qquad (3.17)$$

Since

$$\lim_{L\to\infty}\left(\sup_{\xi\in[0,T]}\int_0^{\xi}\tau^{\gamma}e^{-L(\xi-\tau)}d\tau\right) = 0, \quad (\gamma > -1),$$

there exists $L_0 > 0$ satisfying

$$\sup_{\xi\in[0,T]}\int_0^{\xi}\tau^{\gamma_1}e^{-L(\xi-\tau)}d\tau < \frac{1}{4C_1} \quad \forall L \geq L_0. \qquad (3.18)$$

On the other hand, from (3.15) it implies $\gamma_{L_0}(\{g_n : n \geq 1\}) \geq \gamma_{L_0}(\{v_n : n \geq 1\})$. Hence, combining with (3.17) and (3.18) we get $\gamma_{L_0}(\{v_n : n \geq 1\}) = 0$. So $\beta(\{v_n(t,.)\}) = 0$ for all $t \in [0,T]$. From the conditions (Hc) and (Hd) it implies that $\{f_n\}$ is semicompact. Applying Lemma 3.2 we deduce that $\{g_n : n \geq 1\}$ is relatively compact, so $\nu_{L_0}(D) = (0,0)$ (zero of $\mathbb{R}^2$). The proof is completed. □

3.3 Existence and compactness

In this subsection, we shall establish the compact property of the mild solutions set, denoted by $\mathscr{S}_h^F[0,T]$, of the inclusion (1.1).

Theorem 3.7. *Assume that F satistied the condition (H). Then, $\mathscr{S}_h^F[0,T]$ is a nonempty and compact subset of $\mathcal{C}([0,T];\mathbb{H})$.*

Proof. We consider the multi-valued mapping $\mathcal{M}:\mathcal{C}([0,T];\mathbb{H})\to\mathcal{P}(\mathcal{C}([0,T];\mathbb{H}))$ defined by

$$\mathcal{M}(u) := \{v\in\mathcal{C}([0,T];\mathbb{H}) : v(t,.)=\Phi(f)(t,.), f\in\mathscr{S}_F(u)\}$$

Choose C_1 satisfying

$$\|\Phi(f)(t)\|_{\mathbb{H}} \le C_1\int_0^t \|f(s)\|_{\mathbb{H}}ds. \tag{3.19}$$

and L_0 satisfying (3.18). Applying Lemma 3.5 and Lemma 3.6 we derive that $\mathcal{M}$ is usc and ν_{L_0}-condensing. We define the weighted space

$$\mathcal{C}_{L_0}([0,T];\mathbb{H}) = \left\{v\in\mathcal{C}([0,T];\mathbb{H}) : \exists K>0, \|v(t,.)\|_{\mathbb{H}}\le Ke^{L_0t}\ \forall t\in[0,T]\right\},$$

endowed with norm

$$\|v\|_{\mathcal{C}_{L_0}([0,T];\mathbb{H})} = \sup_{t\in[0,T]} e^{-L_0t}\|v(t,\cdot)\|_{\mathbb{H}}\quad \forall v\in\mathcal{C}_{L_0}([0,T];\mathbb{H}).$$

In the space $\mathcal{C}_{L_0}([0,T];\mathbb{H})$, we denote

$$\overline{B}(r) = \{x\in\mathcal{C}_{L_0}([0,T];\mathbb{H}) : \|x\|_{\mathcal{C}_{L_0}}([0,T],\mathbb{H})\le r\}.$$

Choose $r>(r+1)/4$. Let $u\in\overline{B}(r), f\in\mathscr{S}_F(u)$, $v\in\mathcal{M}(u)$. Using the condition (Hc) we have

$$\begin{aligned}
e^{-L_0t}\|v(t,.)\|_{\mathbb{H}} &= e^{-L_0t}\left\|\Phi(f)(t,.)\right\|_{\mathbb{H}}\\
&\le C_1\int_0^t e^{-L_0(t-s)}e^{-L_0s}s^{\gamma_1}\left(1+\|u(s,.)\|_{\mathbb{H}}\right)ds\\
&\le C_1\int_0^t s^{\gamma_1}(e^{-L_0s}+r)e^{-L_0(t-s)}ds\\
&\le C_1\left((1+r)\int_0^t s^{\gamma_1}e^{-L_0(t-s)}ds\right) < r.
\end{aligned}$$

This implies $v\in\overline{B}(r)$. It follows that $\mathscr{S}_h^F[0,T]\ne\emptyset$ by applying Lemma 2.4. To prove that $\mathscr{S}_h^F[0,T]$ is compact. This is argued similarly to the last part in the proof of the previous theorem. □

3.4 Continuous dependence on parameters

In this part, we look at the relationship between the solution of the parameterized equation (1.6) and a scalar μ in the metric space (E,d). We remind readers of the equation for their convenience.

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t),\mu), & (t,x) \in (0,T] \times \Omega, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, & x \in \Omega. \end{cases} \tag{3.20}$$

Fixed $\mu_0 \in E$, we consider the continuous of respectively mild solutions set, i.e, if μ near enough μ_0, the solution-set corresponding to μ is contained in neighbourhood of the solution-sets corresponding to μ_0.

We consider the continuous dependence on parameters under the assumptions (H_μ).
Let $F : [0,T] \times \mathbb{H} \times E \to KV(\mathbb{H})$ be a mapping satisfying the following conditions:

H_μ(a) : The multi-valued mapping $F(.,u,\mu)$ has a strongly measurable selection for all $(u,\mu) \in \mathbb{H} \times E$;

H_μ(b) : The multi-valued mapping $F(t,.,.) : \mathbb{H} \times E \to KV(\mathbb{H})$ is usc for a.e. $t \in [0,T]$;

H_μ(c) : There is a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u,\mu)||| := \sup_{v \in F(t,u,\mu)} \|v\|_{\mathbb{H}} \leq \alpha(t)(1 + \|u\|_{\mathbb{H}}) \text{ for a.e. } t \in (0,T),$$

for all $u \in \mathbb{H}, \mu \in E$.

H_μ(d) : There exists $\mathcal{B} \in L^1((0,T);\mathbb{R})$ satisfying

$$\beta(F(t,G,E)) \leq \mathcal{B}(t)\beta(G) \text{ for a.e. } t \in (0,T) \text{ for all } G \in b(\mathbb{H}),$$

here β is MNC in $\mathbb{H}$ defined

$$\beta(G) = \inf\{\varepsilon > 0 : G \text{ has a finite } \varepsilon\text{-net}\}. \tag{3.21}$$

For every $(u,\mu) \in \mathcal{C}([0,T];\mathbb{H}) \times E$ we denote

$$\mathscr{S}_{F,\mu}(u) = \left\{ f \in L^1((0,T);\mathbb{H}) | f(t,.) \in F(t,u,\mu), \text{ for a.e. } t \in (0,T) \right\}$$

For every $\mu \in E$, similarly as Theorem 3.7 we also denote multioperator $\mathcal{M}_\mu : \mathcal{C}([0,T];\mathbb{H}) \to \mathcal{P}(\mathcal{C}([0,T];\mathbb{H}))$ defined by

$$\mathcal{M}_\mu(u) := \{v \in \mathcal{C}([0,T];\mathbb{H}) : v(t,.) = \Phi(f)(t,.), f \in \mathscr{S}_{F,\mu}(u)\}.$$

Denote by $\mathscr{H}_h^{F,\mu}$ the family of all local mild solutions of Problem (1.6), i.e. , $u \in \mathscr{H}_h^{F,\mu}$ iff there exist $\tau \in (0,T]$ and $u \in \mathcal{C}([0,T];\mathbb{H})$ such that for all $\overline{\tau} \in [0,\tau]$ and $v_{\overline{\tau}} = u|_{[0,\overline{\tau}]}$, it holds

$$v_{\overline{\tau}} \in \{w \in \mathcal{C}([0,\overline{\tau}];\mathbb{H}) : w(t) = \Phi(f)(t), f \in \mathscr{S}_{F,\mu}(u)\},$$

and $\mathscr{H}_h^{F,\mu}[0,T] := \left\{v \in \mathscr{H}_h^{F,\mu} : v \in \mathcal{M}_\mu(v)\right\}$, here

$$\mathcal{M}_\mu(u) := \{v \in \mathcal{C}([0,T];\mathbb{H}) : v(t) = \Phi(f)(t), f \in \mathscr{S}_{F,\mu}(u)\}.$$

Theorem 3.8. *Assume the condition* (H_μ)*, the set* $\mathscr{H}_h^{F,\mu_0}[0,T]$ *is bounded for some* $\mu_0 \in E$ *and*

$$\mathscr{H}_h^{F,\mu_0}[0,\overline{\tau}] = \mathscr{H}_h^{F,\mu_0}[0,T]|_{[0,\overline{\tau})} \quad \text{for all } \overline{\tau} \in (0,T]. \tag{3.22}$$

Then, for every $\epsilon > 0$*, we can find* $\delta_\epsilon > 0$ *satisfying*

$$\mathscr{H}_h^{F,\mu}[0,T] \subset \mathcal{L}_\epsilon\left(\mathscr{H}_h^{F,\mu_0}[0,T]\right) \quad \text{for all } \lambda \in \mathcal{B}_{\delta_\epsilon}(\mu_0).$$

Proof. Suppose $r > 0$ with $|||\mathscr{H}_h^{F,\mu}[0,T]||| < r$. We will first use the contraction argument to prove the following statement: There exists $\delta > 0$ such that $\mu \in \mathcal{L}_\delta(\mu_0) \subset E$ implies

$$|||\mathscr{H}_h^{F,\mu}(t)||| \le 3r \quad \text{for all } t \in [0,T]. \tag{3.23}$$

Indeed, assume that (3.23) fails. Then, we can choose sequences $\{\mu_n\} \subset E$, $\{t_n\} \subset [0,T]$, $\{u_n\} \subset \mathcal{C}([0,T];\mathbb{H})$, $\mu_n \to \mu_0$ such that $w_n \in \mathcal{M}_{\mu_n}(w_n)$ and

$$\operatorname{dist}\left(w_n(t_n), \mathscr{H}_h^{F,\mu_0}(t_n)\right) \ge 2r, \quad \operatorname{dist}\left(w_n(t), \mathscr{H}_h^{F,\mu_0}(t)\right) < 2r \tag{3.24}$$

for all $t \in [0,t_n)$.

Denote $t_* = \underline{\lim}\{t_n\}$ we will show that $t_* \in (0,T]$. Indeed, suppose $t_* = 0$. Let us have a look at a subsequence of $\{t_n\}$ converging to 0, we also denote this subsequebce by $\{t_n\}$ for convenience. Since $\mathscr{H}_h^{F,\mu_0}$ is bounded and from (3.22) it follows that $\mathscr{H}_h^{F,\mu_0}$ is compact, and so the distance between h and $\mathscr{H}_h^{F,\mu_0}(t_n)$ converges to 0. We get

$$\begin{aligned} 2r &\le \operatorname{dist}\left(w_n(t_n), \mathscr{H}_h^{F,\mu_0}(t_n)\right) \\ &\le \|w_n(t_n) - h\|_{\mathbb{H}} + \operatorname{dist}\left(h, \mathscr{H}_h^{F,\mu_0}(t_n)\right) \\ &\le \left\|\sum_{j=1}^{\infty} e^{\mu_j(T-t_n)}\langle h,\phi_j\rangle\phi_j - h\right\|_{\mathbb{H}} + \|\Phi(f_n)(t_n)\|_{\mathbb{H}} + \operatorname{dist}\left(h, \mathscr{H}_h^{F,\mu_0}(t_n)\right), \end{aligned} \tag{3.25}$$

here $f_n \in \mathscr{S}_{F,\mu_0}(w_n)$ for all $n \in \dot{\mathbb{N}}$. When $n \to \infty$ in (3.25) we obtain $2r \leq 0$. This is a contradiction, we deduce $t_* > 0$.

By the definition of t_*, we can find a number γ with $0 < \gamma < t_* \leq T$ such that w_n are defined on $[0, t_* - \gamma]$ for all n. We next show that for every w_n, there is $\tau_n \in [0, t_* - \gamma] \subsetneq [0, T]$ satisfying

$$\operatorname{dist}\big(w_n(\tau_n), \mathscr{H}_h^{F,\mu_0}(\tau_n)\big) \geq \epsilon. \tag{3.26}$$

For every n, let any $t_\dagger \in [0, t_n)$, by the compactness of $\mathscr{H}_h^{F,\mu_0}$ we suppose that $\|w_n(t) - w_\dagger(t)\|_{\mathbb{H}} < \epsilon$ for some $w_\dagger \in \mathscr{H}_h^{F,\mu_0}$. We get

$$\begin{aligned}&\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathbb{H}}\\&\qquad \leq \|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathbb{H}} + \|w_\dagger(t_\dagger + t) - w_\dagger(t_\dagger)\|_{\mathbb{H}} + \|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathbb{H}}.\end{aligned}$$

By the same argument as in the proof of Lemma 3.2, we can select t is small enough such that both $\|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathbb{H}}$ and $\|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathbb{H}}$ are less than $\frac{\epsilon}{4}$. Therefore, $\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathbb{H}} \leq 3\epsilon/2$, this contradicts (3.24). Namely, (3.26) is proved.

Now, by similar arguments as proving Lemma 3.6, we note that $\mathcal{M}_* : E \times \mathcal{C}([0, t_* - \gamma]; \mathbb{H}) \to KV(C([0, t_* - \gamma]; \mathbb{H}))$, $\mathcal{M}_*(\mu, u) = \mathcal{M}^\mu(u)$, is ν_L-condensing for some $L > 0$. That ensures the relative compactness of $\{w_n|_{[0,t_*-\gamma]}\}$. Let us take $w_* = \lim w_n|_{[0,\gamma-t_*]}$, which belongs to $\mathcal{M}_*(\lambda_0, w_*)$ on $[0, t_* - \gamma]$. So, by passing to the limit in (3.26) we get

$$\operatorname{dist}\big(w_*(t_*), \mathscr{H}_h^{F,\mu_0}(t_*)\big) \geq \epsilon.$$

Hence, the solution u_* cannot be extended to the interval $[0, T]$, this contradicts (3.22). We complete the proof by applying Lemma 2.6. □

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