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General decay and lifespan of solutions for a system of nonlinear pseudoparabolic equations with viscoelastic term

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ABSTRACT

In this paper, we consider the Robin-Dirichlet problem for a system of nonlinear pseudoparabolic equations with viscoelastic term. By the Faedo-Galerkin method, we first prove the existence and uniqueness of solution for the problem. Next, we give a sufficient condition to get the global existence and decay of the weak solution. Finally, by the concavity method, we prove the blow-up result of the solution when the initial energy is negative. Furthermore, we establish here the lifespan of the solution by finding the upper bound and the lower bound for the blow-up time.

Keywords: *Nonlinear pseudoparabolic equations; Faedo-Galerkin method; Local existence; Blow-up; Lifespan; The global existence and decay of weak solutions.*

1 Introduction

In this paper, we consider the initial-boundary value problem for the system of nonlinear pseudoparabolic equations with the Robin-Dirichlet boundary conditions as follows

$$\begin{cases} u_t - \lambda_1 u_{txx} - \dfrac{\partial}{\partial x}\left(\mu_1(x,t)u_x\right) = f_1(x,t,u,v,u_x,v_x),\ 0<x<1,\ 0<t<T, \\ v_t - \lambda_2 v_{txx} - \dfrac{\partial}{\partial x}\left(\mu_2(x,t)v_x\right) + \displaystyle\int_0^t g(t-s)v_{xx}(x,s)ds \\ \qquad\qquad\qquad\qquad = f_2(x,t,u,v,u_x,v_x),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0, \\ \left(u(x,0), v(x,0)\right) = \left(\tilde{u}_0(x), \tilde{v}_0(x)\right), \end{cases} \tag{1.1}$$

where $\zeta \geq 0$, λ_1, $\lambda_2 > 0$ are given constants and g, μ_i, f_i, $(i=1,2)$, $\tilde{u}_0$, $\tilde{v}_0$ are given functions satisfying conditions specified later.

The pseudoparabolic equation

$$u_t - u_{xxt} = F\left(x,t,u,u_x,u_{xx},u_{xt}\right),\ 0<x<1,\ t>0, \tag{1.2}$$

with the initial condition $u(x,0) = \tilde{u}_0(x)$ and different boundary conditions has been extensively studied by many authors, see for example [3], [8], [10]- [15], [18], [20], [22] among others and the references given therein. In these works, numerous interesting results related to existence, asymptotic behavior, blowup, and decay of solutions were obtained.

An important special case of the model (1.2) is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.3}$$

it was studied by Amick et al. [1] with $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geq 0$, in which the solution of (1.3) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm *as* $t \to +\infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0,1]$, $t \geq 0$, the model has the form (1.3) was also investigated earlier by Bona and Dougalis [2], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geq 0$ and on $\alpha > 0$. The results obtained in [1] were developed by many authors, such as by Zhang [23] for equations of the form

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0, \tag{1.4}$$

where $m \geq 0$, see Meyvaci [8].

The linear version of (1.2) was first studied by S.L. Sobolev [17] in 1954. Therefore, the equation of the form (1.2) is also called a Sobolev type equation. Mathematical study of pseudo-parabolic equations goes back to works of Showalter (see [14]- [16]) in the seventies, since then, numerous of interesting results about linear and nonlinear pseudoparabolic equations have been obtained. It is also well known that the work [16] is the first paper on nonlinear pseudoparabolic equation. These equations appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory, see M. Meyvaci [8] and the references given therein.

The problem (1.1) is a kind of viscoelastic pseudoparabolic problem, the Volterra integral in the second equation of (1.1) is a memory term, also known as called the viscoelastic term, is the cause of viscoelastic damping. In recent years, much attention has been paid to pseudo-parabolic equations with memory terms or viscoelastic terms. For instance, Shang and Guo [13] proved the existence, uniqueness, regularities of the global strong solution and gave some conditions of the nonexistence of global solution for the nonlinear pseudoparabolic equation with Volterra integral term

$$\begin{aligned} u_t - f(u)_{xx} - u_{xxt} - \int_0^t \lambda(t-s) \left(\sigma\left(u(x,s), u_x(x,s)\right)\right)_x ds = f\left(x,t,u,u_x\right), \\ 0 < x < 1,\ t > 0. \end{aligned} \tag{1.5}$$

In [18], Sun et. al. considered the Dirichlet problem for the nonlinear pseudoparabolic equation with a power source term and a memory term as follows

$$\begin{cases} u_t - \Delta u - \Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau) d\tau = |u|^{p-2}\, u, \text{ in } \Omega \times (0,T), \\ u = 0, \text{ on } \partial\Omega \times (0,T), \\ u(0) = u_0, \text{ in } \Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, $p > 2$, $T \in (0,\infty]$, $u_0 \in H^1(\Omega)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nonincreasing function. The authors used the Levine's classical concavity method and the improved potential well method to obtain the global existence and the finite time blow-up phenomena of solutions.

Recently, in [10], the following initial boundary problem for a nonlinear pseudoparabolic equation containing a viscoelastic term

$$\begin{cases} u_t - \left(\mu\left(t\right) + \alpha\left(t\right) \dfrac{\partial}{\partial t}\right) \left(u_{xx} + \dfrac{1}{x} u_x\right) + \displaystyle\int_0^t g\left(t-s\right) \left(u_{xx}(s) + \frac{1}{x} u_x(s)\right) ds \\ \qquad = f\left(x,t,u\right),\ 1 < x < R,\ t > 0, \\ u_x(1,t) - \zeta u(1,t) = u(R,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \end{cases} \tag{1.7}$$

has been considered, and the results of existence, uniqueness, blow-up and decay estimates of the solution for (1.7) have been proved. Furthermore, the authors also established the lifespan for the solution via finding the upper bound and the lower bound for the blow-up times. Generalizing the results in [10], the authors in [12] studied general decay and blow-up of solutions for the following pseudoparabolic nonlinear equation of Kirchhoff-Carrier type with viscoelastic term

$$\begin{aligned} u_t - \left(\mu\left(t, \|u(t)\|_0^2, \|u(t)\|_a^2\right) + \alpha(t)\frac{\partial}{\partial t}\right)(u_{xx} + \frac{1}{x}u_x) \\ + \int_0^t g(t-s)(u_{xx}(s) + \frac{1}{x}u_x(s))ds \\ = f(x,t,u,u_x,u_t), \ 1 < x < R, \ t > 0, \end{aligned} \tag{1.8}$$

associated with the initial condition and the nonlocal boundary condition $(1.7)_{2,3}$, in which $\|u(t)\|_0^2 = \int_1^R xu^2(x,t)dx$, $\|u(t)\|_a^2 = \int_1^R xu_x^2(x,t)dx + \zeta u^2(1,t)$.

At the present time, to the best of our knowledge, there are many publications on properties of solutions to single parabolic/pseudoparabolic equations, but it seems that there are relatively few results for systems of these types. We refer to some results as in [3], [4]- [6] and the references therein. And recently in [11], Ngoc et. al. have also considered the initial-boundary value problem for the system of nonlinear pseudoparabolic equations with Robin-Dirichlet conditions and established here the existence, uniqueness, blow-up and general decay of solutions.

Inspired and motivated by the idea of the above mentioned works, we study the existence, uniqueness, blow-up and general decay of solutions for Prob. (1.1). This paper consists of three sections. Section 2 presents preliminaries and Section 3 presents the main results. In Subsection 3.1, by using the linear approximating method together with the Galerkin method, we establish the local existence and uniqueness of a weak solution. Subsection 3.2, we consider Prob. (1.1) with $f_i(x,t,u,v,u_x,v_x) = f_i(u,v) + F_i(x,t)$, $i = 1, 2$, and prove a sufficient condition for the global existence and decay of solution via the energy method. Finally, Subsection 3.3 is devoted to the study of the blow-up property for Prob. (1.1) in the special case $f_i(x,t,u,v,u_x,v_x) = f_i(u,v)$, $i = 1, 2$. Based on the concavity method and the improved potential method, we describe the blow-up phenomenon of solution with negative initial energy.This section also derives the lifespan for solution via finding the upper bound and the lower bound for the blow-up time. The results obtained here relatively generalize the results in [10]- [12] and those are based on applying the methods and technics in [9], [11].

2 Preliminaries

First, we put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ \underset{0<t<T}{ess\sup} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

Denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x,t)$, $u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t) = \frac{\partial u}{\partial x}(x,t)$,

$u_{xx}(t) = \dfrac{\partial^2 u}{\partial x^2}(x,t).$

With $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^4)$, $f = f(x,t,y_1,\cdots,y_4)$, we put $D_1 f = \dfrac{\partial f}{\partial x}$, $D_2 f = \dfrac{\partial f}{\partial t}$, $D_{i+2} f = \dfrac{\partial f}{\partial y_i}$, $i = 1,\cdots,4$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_6^{\alpha_6} f$, $\alpha = (\alpha_1,\cdots,\alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 \le k$, $D^{(0,\cdots,0)} f = D^0 f = f$.

Similarly, with $\mu \in C^k([0,1] \times [0,T^*])$, $\mu = \mu(x,t)$, we set $D_1\mu = \dfrac{\partial \mu}{\partial x}$, $D_2\mu = \dfrac{\partial \mu}{\partial t}$, và $D^\beta \mu = D_1^{\beta_1} D_2^{\beta_2} \mu$, $\beta = (\beta_1,\beta_2) \in \mathbb{Z}_+^2$, $|\beta| = \beta_1 + \beta_2 \le k$, $D^{(0,0)}\mu = \mu$.

On H^1, we use the norm $\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}}$, and we define the following closed subspace V of H^1

$$V = \{v \in H^1 : v(1) = 0\}. \tag{2.1}$$

Then, we have the following standard lemmas concerning the imbeddings of H^1 into $C^0(\bar{\Omega})$ and of V into $C^0(\bar{\Omega})$.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\bar{\Omega})$ is compact, and*

$$\|v\|_{C^0(\bar{\Omega})} \le \sqrt{2}\|v\|_{H^1},\ \forall v \in H^1.$$

Lemma 2.2. *The imbedding $V \hookrightarrow C^0(\bar{\Omega})$ is compact. Moreover, we have*

$$\begin{array}{ll} (i) & \|v\|_{C^0(\overline{\Omega})} \le \|v_x\|,\ \forall v \in V, \\ (ii) & \frac{1}{\sqrt{2}}\|v\|_{H^1} \le \|v_x\| \le \|v\|_{H^1},\ \forall v \in V. \end{array}$$

For $T^* > 0$, let $\zeta \ge 0$ and $\mu_1 \in C^1\left(\bar{\Omega} \times [0,T^*]\right)$, $\mu_1' \in C^0\left(\bar{\Omega} \times [0,T^*]\right)$. On $V \times V$, we consider the symmetric bilinear forms $a(\cdot,\cdot)$, and the familles of symmetric bilinear forms $\{\bar{a}(t;\cdot,\cdot)\}_{t\in[0,T]}$, $\{\bar{a}'(t;\cdot,\cdot)\}_{t\in[0,T]}$ defined by

$$\begin{aligned} a(u,\varphi) &= \langle u_x, \varphi_x\rangle + \zeta u(0)\varphi(0), \\ \bar{a}(t;u,\varphi) &= \langle \mu_1(t)\, u_x, \varphi_x\rangle + \zeta\mu_1(0,t)\, u(0)\, \varphi(0), \\ \bar{a}'(t;u,\varphi) &= \langle \mu_1'(t)\, u_x, \varphi_x\rangle + \zeta\mu_1'(0,t)\, u(0)\, \varphi(0),\ \forall (u,\varphi) \in V \times V,\ t \in [0,T^*]. \end{aligned} \tag{2.2}$$

Then we can prove the following properties without difficulty.

Lemma 2.3. *Let $\zeta \ge 0$ and $\mu_1,\ \mu_1' \in C^0\left(\bar{\Omega} \times [0,T]\right)$ such that $\mu_1(x,t) \ge \mu_{i*} > 0$ for all $(x,t) \in \bar{\Omega} \times [0,T]$. Then,*
(i) The symmetric bilinear form $a(\cdot,\cdot)$, and the family of symmetric bilinear form $\{\bar{a}(t;\cdot,\cdot)\}_{t\in[0,T]}$ defined by (2.2) are continuous on $V \times V$ and coercive in V.
(ii) The family of symmetric bilinear form $\{\bar{a}'(t;\cdot,\cdot)\}_{t\in[0,T^*]}$ defined by (2.2) are continuous on $V \times V$. Moreover, we have

$$\begin{aligned} a(u,u) &\ge \|u_x\|^2,\ \forall u \in V, \\ |a(u,\varphi)| &\le (1+\zeta)\|u_x\|\|\varphi_x\|,\ \forall u,\varphi \in V, \\ \bar{a}(t;u,u) &\ge \mu_{1*}\|u\|_a^2 \ge \mu_{1*}\|u_x\|^2,\ \forall u \in V,\ t \in [0,T^*], \\ |\bar{a}(t;u,\varphi)| &\le \|\mu_1\|_{C^0(\bar{\Omega}\times[0,T^*])}\|u\|_a\|\varphi\|_a \\ &\le \|\mu_1\|_{C^0(\bar{\Omega}\times[0,T^*])}(1+\zeta)\|u_x\|\|\varphi_x\|,\ \forall u,\varphi \in V,\ t \in [0,T^*], \\ |\bar{a}'(t;u,\varphi)| &\le \|\mu_1'\|_{C^0(\bar{\Omega}\times[0,T^*])}\|u\|_a\|\varphi\|_a \\ &\le \|\mu_1'\|_{C^0(\bar{\Omega}\times[0,T^*])}(1+\zeta)\|u_x\|\|\varphi_x\|,\ \forall u,\varphi \in V,\ t \in [0,T^*], \end{aligned}$$

in which

$$\|u\|_a = \sqrt{a(u,u)},\ \forall u \in V.$$

3 Main results

3.1 The existence and uniqueness of a weak solution

Consider $T^* > 0$ fixed. We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{v}_0) \in (H^2 \cap V) \times (H^2 \cap H_0^1)$, $\tilde{u}_{0x}(0) - \zeta \tilde{u}_0(0) = 0$;

(H_2) $\mu_1,\ \mu_2 \in C^2([0,1] \times [0,T^*])$, and there exist the positive constants $\mu_{1*},\ \mu_{2*}$ such that $\mu_i(x,t) \geq \mu_{i*},\ \forall (x,t) \in [0,1] \times [0,T^*],\ i = 1,2$;

(H_3) $g \in H^1(0,T^*)$;

(H_4) $f_i \in C^1\left([0,1] \times [0,T^*] \times \mathbb{R}^4\right),\ i = 1,2$, such that $f_1(1,t,0,0,y_3,y_4) = f_2(1,t,0,0,y_3,y_4) = f_2(0,t,y_1,0,y_3,y_4) = 0$, $\forall t \in [0,T^*],\ \forall y = (y_1,y_3,y_4) \in \mathbb{R}^3$.

For each $T \in (0,T^*]$, we denote

$$\begin{aligned} W_T = \{(u,v) \in L^\infty\left(0,T;(H^2 \cap V) \times (H^2 \cap H_0^1)\right) :& \\ (u',v') \in L^\infty\left(0,T;(H^2 \cap V) \times (H^2 \cap H_0^1)\right)\},& \\ W_1(T) = \left\{(u,v) \in C^0([0,T];V \times H_0^1) : (u',v') \in L^2\left(0,T;V \times H_0^1\right)\right\},& \end{aligned} \tag{3.1}$$

are Banach spaces (see Lions [7]) with respect to the norms

$$\begin{aligned} &\|(u,v)\|_{W_T} \\ &= \max\left\{\|(u,v)\|_{L^\infty\left(0,T;(H^2\cap V)\times(H^2\cap H_0^1)\right)}, \left\|(u',v')\right\|_{L^\infty\left(0,T;(H^2\cap V)\times(H^2\cap H_0^1)\right)}\right\}, \\ &\|(u,v)\|_{W_1(T)} = \|(u,v)\|_{C^0([0,T];V\times H_0^1)} + \left\|(u',v')\right\|_{L^2\left(0,T;V\times H_0^1\right)}. \end{aligned} \tag{3.2}$$

Definition 3.1. For each $T \in (0,T^*]$, a couple of functions $(u,v) \in W_T$ is called a weak solution of Prob. (1.1) if and only if (u,v) satisfies the following variational problem

$$\begin{cases} \langle u'(t),\varphi\rangle + \lambda_1 a(u'(t),\varphi) + \bar{a}(t;u(t),\varphi) = \langle f_1[u,v](t),\varphi\rangle, \\ \langle v'(t),\psi\rangle + \lambda_2\langle v_x'(t),\psi_x\rangle + \langle \mu_2(t)\,v_x(t),\psi_x\rangle = \displaystyle\int_0^t g\,(t-s)\,\langle v_x(s),\psi_x\rangle ds + \\ \langle f_2[u,v](t),\psi\rangle, \end{cases} \tag{3.3}$$

for all $(\varphi,\psi) \in V \times H_0^1$, and a.e. $t \in (0,T)$, together with the initial condition

$$(u(0),v(0)) = (\tilde{u}_0,\tilde{v}_0), \tag{3.4}$$

where

$$f_i[u,v](x,t) = f_i(x,t,u(x,t),v(x,t),u_x(x,t),v_x(x,t)),\ \ i = 1,2. \tag{3.5}$$

For each $M > 0$ given, we set the constant

$$B_T(M) = \{(u,v) \in W_T : \|(u,v)\|_{W_T} \leq M\}. \tag{3.6}$$

Now, we establish the following recurrent sequence $\{(u_m,v_m)\}$. The first term is chosen as $(u_0,v_0) \equiv (0,0)$, suppose that

$$(u_{m-1},v_{m-1}) \in B_T(M). \tag{3.7}$$

Find $(u_m,v_m) \in B_T(M)$ $(m \geq 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m'(t),\varphi\rangle + \lambda_1 a(u_m'(t),\varphi) + \bar{a}(t;u_m(t),\varphi) = \langle F_{1m}(t),\varphi\rangle, \\ \langle v_m'(t),\psi\rangle + \lambda_2\langle v_{mx}'(t),\psi_x\rangle + \langle \mu_2\,(t)\,v_{mx}(t),\psi_x\rangle \\ \quad = \displaystyle\int_0^t g\,(t-s)\,\langle v_{mx}(s),\psi_x\rangle ds + \langle F_{2m}(t),\psi\rangle,\ \forall(\varphi,\psi) \in V \times H_0^1,\ a.e.\ t \in (0,T), \\ (u_m(0),v_m(0)) = (\tilde{u}_0,\tilde{v}_0), \end{cases} \tag{3.8}$$

in which

$$F_{im}(x,t) = f_i[u_{m-1}, v_{m-1}](x,t),\ i = 1,2. \tag{3.9}$$

Then we have the following theorem.

Theorem 3.2. *Let $(H_1) - (H_4)$ hold. Then there exist constants M, $T > 0$ such that, for $(u_0, v_0) \equiv (0,0)$, there exists a recurrent sequence $\{(u_m, v_m)\} \subset B_T(M)$ defined by (3.7)-(3.9).*

Proof. The proof of Theorem 3.2 is based on the Faedo-Galerkin approximation (introduced by Lions [7]) associated with a priori estimate, thereby deriving weakly converging subsequences in appropriate function spaces via the compact imbedding theorems. Here, the Banach's contraction principle is also used to prove the existence of a Faedo-Galerkin approximation solution. □

The convergence of the recurrent sequence $\{(u_m, v_m)\}$ to the weak solution (u,v) of Prob. (1.1) is given by the following theorem, whose proof can be referenced in [11].

Theorem 3.3. *Suppose that $(H_1) - (H_4)$ are satisfied. Then, the recurrent sequence $\{(u_m, v_m)\}$ defined by (3.7)-(3.9) converges strongly to a couple of functions (u,v) in $W_1(T)$ and $(u,v) \in B_T(M)$ is the unique weak solution of Prob. (1.1). Moreover, we have the following estimate*

$$\|(u_m, v_m) - (u,v)\|_{W_1(T)} \le C_T k_T^m,\ \textit{for all } m \in \mathbb{N}, \tag{3.10}$$

where $k_T \in [0,1)$ and C_T is a constant depending only on T, f_1, f_2, g, μ_1, μ_2, $\tilde{u}_0$, $\tilde{v}_0$ and k_T.

3.2 General decay of the solution

In this subsection, Prob. (1.1) is considered in the form

$$\begin{cases} u_t - \lambda_1 u_{txx} - \dfrac{\partial}{\partial x}\left(\mu_1(x,t)u_x\right) = f_1(u,v) + F_1(x,t),\ 0 < x < 1,\ t > 0, \\ v_t - \lambda_2 v_{txx} - \dfrac{\partial}{\partial x}\left(\mu_2(x,t)v_x\right) + \displaystyle\int_0^t g(t-s)v_{xx}(x,s)ds \\ \qquad\qquad\qquad\qquad\qquad = f_2(u,v) + F_2(x,t),\ 0 < x < 1,\ t > 0, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0, \\ (u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)), \end{cases} \tag{3.11}$$

where $\zeta \ge 0$; λ_1, $\lambda_2 > 0$ are given constants and $\tilde{u}_0$, $\tilde{v}_0$, g, μ_i, f_i, F_i, $(i = 1,2)$, are given functions satisfying conditions specified later.

3.2.1 Local existence and Uniqueness

Based on the results obtained in Subsection 3.1, we can propose the following assumptions to get the local existence and uniqueness of a weak solution for Prob. (3.11).

$(\tilde{H}_1)$ $(\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1$;

$(\tilde{H}_2)$ μ_1, $\mu_2 \in C^1([0,1] \times \mathbb{R}_+)$, and there exist the positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \ge \mu_{i*}$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$, $i = 1,2$;

$(\tilde{H}_3)$ $g \in C^1(\mathbb{R}_+; \mathbb{R}_+)$;

$(\tilde{H}_4)$ There exist the function $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the positive constants $\bar{d}_2 > 0$, $\alpha > 2$, $\beta > 2$, such that

(i) $\dfrac{\partial \mathcal{F}}{\partial u} = f_1(u,v)$, $\dfrac{\partial \mathcal{F}}{\partial v} = f_2(u,v)$, $\forall (u,v) \in \mathbb{R}^2$,

(ii) $\mathcal{F}(u,v) \le \bar{d}_2\left(1 + |u|^\alpha + |v|^\beta\right)$, $\forall (u,v) \in \mathbb{R}^2$;

$(\tilde{H}_5)$ $F_i \in L^2(\mathbb{R}_+; L^2)$, $i = 1,2$.

Using the standard arguments of density and Theorem 3.3, we obtain the following theorem.

Theorem 3.4. *Let $(\tilde{H}_1)-(\tilde{H}_5)$ hold. Then, there exists $T>0$ such that Prob. (3.11) has a unique solution (u,v) satisfying*

$$(u,v)\in C^0([0,T];V\times H_0^1)\ \textit{và}\ (u',v')\in L^2(0,T;V\times H_0^1).$$

3.2.2 Global existence and general decay of the solution

We strengthen the following assumptions

$(\tilde{H}_1)$ $(\tilde{u}_0,\tilde{v}_0)\in V\times H_0^1$;

(D_2) $\mu_1,\ \mu_2\in C^1([0,1]\times\mathbb{R}_+)$, and there exist the positive constants $\mu_{1*},\ \mu_{2*}$ such that $\mu_i(x,t)\geq\mu_{i*},\ \mu_i'(x,t)\leq 0,\ \forall(x,t)\in[0,1]\times\mathbb{R}_+,\ i=1,2$;

(D_3) $g\in C^1(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)$, and there exists the function $\zeta\in C^1(\mathbb{R}_+)$ such that

(i) $\zeta'(t)\leq 0<\zeta(t),\ \forall t\geq 0,\ \int_0^\infty\zeta(t)dt=+\infty$,

(ii) $g'(t)\leq-\zeta(t)g(t),\ 0<g(t)\leq g(0),\ \forall t\geq 0$,

(iii) $L_*\equiv\mu_{2*}-\int_0^\infty g(s)ds>0$;

(D_4) There exists the function $\mathcal{F}\in C^2(\mathbb{R}^2;\mathbb{R})$ and the positive constants $d_2>p,\ \bar{d}_2>0,\ q_i>2,\ \bar{q}_i>2\ (i=1,\cdots,N)$, such that

(i) $D_1\mathcal{F}(u,v)=f_1(u,v),\ D_2\mathcal{F}(u,v)=f_2(u,v),\ \forall(u,v)\in\mathbb{R}^2$,

(ii) $uf_1(u,v)+vf_2(u,v)\leq d_2\mathcal{F}(u,v),\ \forall(u,v)\in\mathbb{R}^2$,

(iii) $\mathcal{F}(u,v)\leq\bar{d}_2\sum_{i=1}^N\left(|u|^{q_1}+|v|^{\bar{q}_1}\right),\ \forall(u,v)\in\mathbb{R}^2$;

(D_5) $F_i\in L^2(\mathbb{R}_+;L^2)$ such that there exist two constants $\bar{C}_0>0,\ \bar{\gamma}_0>0$, satisfying $\|F_1(t)\|^2+\|F_2(t)\|^2\leq\bar{C}_0e^{-\bar{\gamma}_0t},\ \forall t\geq 0$;

(D_6) $p>\max\{2,d_2\}$, and $0<\bar{g}(\infty)<\dfrac{d_2\eta_*}{p-d_2}+\mu_{2*}$.

Remark 3.5. *We give an example of the functions $g(t),\ f_1(u,v),\ f_2(u,v)$ satisfying (D_3), (D_4) as below*

$$\begin{aligned}g(t)&=\sigma\exp\left(-\int_0^t\zeta(s)ds\right),\\ f_1(u,v)&=\alpha k_1|u|^{\alpha-2}u+\alpha_1k_3|u|^{\alpha_1-2}u|v|^{\beta_1},\\ f_2(u,v)&=\beta k_2|v|^{\beta-2}v+\beta_1k_3|u|^{\alpha_1}|v|^{\beta_1-2}v,\end{aligned}$$

where $\sigma,\ k_1,\ k_2,\ k_3>0,\ \alpha,\ \beta,\ \alpha_1,\ \beta_1>2$ are constants, $\zeta\in C^1(\mathbb{R}_+)$ such that $\zeta'(t)\leq 0<\zeta(t),\ \forall t\geq 0$, $\int_0^\infty\zeta(t)dt=+\infty$.

Now, we consider the Lyapunov functional defined as follows

$$\mathcal{L}(t)=E(t)+\delta\Psi(t),\ t>0,$$

where $\delta>0$ is chosen later and

$$E(t)=\frac{1}{2}\tilde{E}(t)-\overline{\mathcal{F}}(t)=\frac{1}{2}\tilde{E}(t)-\overline{\mathcal{F}}(t)=\left(\frac{1}{2}-\frac{1}{p}\right)\tilde{E}(t)+\frac{1}{p}I(t),\tag{3.13}$$

$$\Psi(t)=\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|v(t)\|^2+\frac{\lambda_1}{2}\|u(t)\|_a^2+\frac{\lambda_2}{2}\|v_x(t)\|^2,\tag{3.14}$$

in which

$$\tilde{E}(t) = (g_* \Diamond u)(t) + (g * v)(t) + \bar{a}(t; u(t), u(t)) + \left\| \sqrt{\mu_2(t)} v_x(t) \right\|^2 - \bar{g}(t) \left\| v_x(t) \right\|^2, \tag{3.15}$$

$$\begin{aligned} \overline{\mathcal{F}}(t) &= \int_0^1 \mathcal{F}\left(u(x,t), v(x,t)\right) dx, \\ I(t) &= I\left(u(t)\right) = \tilde{E}(t) - p\overline{\mathcal{F}}(t), \\ (g * v)(t) &= \int_0^t g(t-s) \left\| v_x(t) - v_x(s) \right\|^2 ds, \\ \bar{g}(t) &= \int_0^t g(s) ds, \\ (g_* \Diamond u)(t) &= \int_0^t g_*(t-s) \left\| u'(s) \right\|^2 ds, \end{aligned}$$

with $g_*(t) = 2\bar{\lambda}_* e^{-2\bar{k}_* t}$, where $\bar{k}_*$, $\bar{\lambda}_*$ are the positive constants sucth that $\bar{k}_* > 0$, $0 < \bar{\lambda}_* < 1$.

In the following, we prove that if

$I(0) = \bar{a}(0; \tilde{u}_0, \tilde{u}_0) + \left\| \sqrt{\mu_2(0)} \tilde{v}_{0x} \right\|^2 - p \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{u}_0(x)) dx > 0$, and if the initial energy is small enough, then global existence is obtained and the energy of the solution decays as $t \to +\infty$.

First, we estimate $E'(t)$.

Lemma 3.6. *Suppose that* $(\tilde{H}_1)$, $(D_2) - (D_6)$ *hold. Then*

$$\begin{aligned} E'(t) \leq &- \left(1 - \bar{\lambda}_* - \frac{\varepsilon_1}{2}\right) \left(\left\| u'(t) \right\|^2 + \left\| v'(t) \right\|^2 \right) \\ &- \bar{k}_* (g_* \Diamond u)(t) - \frac{1}{2} \zeta(t) (g * v)(t) + \frac{1}{2\varepsilon_1} \rho_1(t), \end{aligned} \tag{3.16}$$

$\forall \varepsilon_1 > 0$, $\forall t > 0$, *in which* $\rho_1(t) = \left\| F_1(t) \right\|^2 + \left\| F_2(t) \right\|^2$.

Proof. Multiplying the equation (3.11) by $(u'(x,t), v'(x,t))$ and integrating on $(0,1)$, we get

$$\begin{aligned} E'(t) = &\frac{1}{2} \left[\bar{a}'(t; u(t), u(t)) + \int_0^1 \mu_2'(x,t) v_x^2(x,t) dx \right] \\ &- (1 - \bar{\lambda}_*) \left\| u'(t) \right\|^2 - \left\| v'(t) \right\|^2 - \lambda_1 \left\| u'(t) \right\|_a^2 - \lambda_2 \left\| v_x'(t) \right\|^2 \\ &- \bar{k}_* (g_* \Diamond u)(t) + \frac{1}{2} (g' * v)(t) - \frac{1}{2} \bar{g}(t) \left\| v_x(t) \right\|^2 + \langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle. \end{aligned} \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} &\bar{a}'(t; u(t), u(t)) + \int_0^1 \mu_2'(x,t) v_x^2(x,t) dx \leq 0, \\ &\langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle \leq \frac{\varepsilon_1}{2} \left(\left\| u'(t) \right\|^2 + \left\| v'(t) \right\|^2 \right) + \frac{1}{2\varepsilon_1} \rho_1(t), \end{aligned} \tag{3.18}$$

$$\frac{1}{2} (g' * v)(t) \leq -\frac{1}{2} \zeta(t) (g * v)(t), \ \forall \varepsilon_1 > 0, \ \forall t > 0.$$

Then, (3.17) and (3.18) lead to (3.16). Lemma 3.6 is proved. □

Next, using Lemma 3.6, we prove the following lemma in order to obtain the global existence.

Lemma 3.7. *Assume that* $(D_2)-(D_6)$ *hold. Let* $(\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1$ *such that* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^* \equiv L_* - p\bar{d}_2 \max\left\{\sum_{i=1}^{N} R_*^{q_i-2}, \sum_{i=1}^{N} R_*^{\bar{q}_i-2}\right\} > 0, \tag{3.19}$$
$$0 < \bar{g}(\infty) = \int_0^\infty g(s)ds < \frac{d_2\eta^*}{p-d_2} + \mu_{2*},$$

with

$$\begin{aligned} L_* &= \min\{\mu_{1*}, \mu_{2*} - \bar{g}(\infty)\} > 0, \\ R_* &= \sqrt{\frac{2pE_*}{(p-2)L_*}}, \\ E_* &= E(0) + \frac{1}{4(1-\bar{\lambda}_*)}\int_0^\infty \left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right) dt. \end{aligned} \tag{3.20}$$

Then $I(t) > 0$, $\forall t \geq 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) = I(u(t), v(t)) > 0, \ \forall t \in [0, T_1]. \tag{3.21}$$

It follows from (3.13), (3.21) that

$$E(t) \geq \frac{p-2}{2p}\left[(g_*\Diamond u)(t) + (g * v)(t) + L_*\left(\|u(t)\|_a^2 + \|v_x(t)\|^2\right)\right], \ \forall t \in [0, T_1]. \tag{3.22}$$

Combining (3.16) in Lemma 3.6 and (3.22), we get

$$\|u(t)\|_a^2 + \|v_x(t)\|^2 \leq \frac{2pE(t)}{(p-2)L_*} \leq \frac{2pE_*}{(p-2)L_*} \equiv R_*^2, \ \forall t \in [0, T_1]. \tag{3.23}$$

Based on $(D_4)_{(iii)}$, (3.23), it gives

$$\begin{aligned} p\overline{\mathcal{F}}(t) &= p\int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx \leq p\bar{d}_2 \sum_{i=1}^{N}\left(\|u(t)\|_{L^{q_i}}^{q_i} + \|v(t)\|_{L^{\bar{q}_i}}^{\bar{q}_i}\right) \\ &\leq p\bar{d}_2 \max\left\{\sum_{i=1}^{N} R_*^{q_i-2}, \sum_{i=1}^{N} R_*^{\bar{q}_i-2}\right\}\left(\|u(t)\|_a^2 + \|v_x(t)\|^2\right). \end{aligned} \tag{3.24}$$

Therefore

$$I(t) \geq (g_*\Diamond u)(t) + (g * v)(t) + \eta^*\left(\|u(t)\|_a^2 + \|v_x(t)\|^2\right) \geq 0, \ \forall t \in [0, T_1]. \tag{3.25}$$

Put $T_\infty = \sup\{T_1 > 0 : I(t) > 0, \ \forall t \in [0, T_1]\}$.

Suppose that $T_\infty < +\infty$, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$.

If $I(T_\infty) = 0$, using the similar argument as in [11], it implies from (3.25) that $I(0) = 0$. This is a contradiction, since $I(0) > 0$. Thus, $I(T_\infty) > 0$.

By the same arguments as above, we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, \tilde{T}_\infty]$. This is a contradiction to the definition of T_∞. Hence, $T_\infty = +\infty$, i.e. $I(t) > 0$, $\forall t \geq 0$. Lemma 3.7 is proved. □

Next, we establish the decay of the solution of (3.11). For this goal, we put

$$E_1(t) = (g_*\Diamond u)(t) + (g * v)(t) + \|u(t)\|_a^2 + \|v_x(t)\|^2 + I(t), \tag{3.26}$$

and we prove two lemmas (Lemmas 3.8, 3.9) below.

Lemma 3.8. *There exist the positive constants β_1, β_2, $\bar{\beta}_1$, $\bar{\beta}_2$ such that*

$$\begin{aligned} &(i)\ \ \beta_1 E_1(t) \leq \mathcal{L}(t) \leq \beta_2 E_1(t),\ \ \forall t \geq 0, \\ &(ii)\ \bar{\beta}_1 E_1(t) \leq E(t) \leq \bar{\beta}_2 E_1(t),\ \ \forall t \geq 0. \end{aligned} \tag{3.27}$$

Proof. The proof of Lemma 3.8 is not difficult, so we omitt it. □

Lemma 3.9. *The functional $\Psi(t)$ satisfies the following estimation*

$$\begin{aligned} \Psi'(t) \leq& \left(\frac{1}{2\varepsilon_3} + \frac{d_2}{p}\right)(g * v)(t) + \frac{d_2}{p}(g_* \diamondsuit u)(t) - \frac{\delta_1 d_2}{p} I(t) \\ &- \left[(1-\delta_1)\frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{1*} - \frac{\varepsilon_2}{2}\right] \|u(t)\|_a^2 \\ &- \left[(1-\delta_1)\frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{2*} - \frac{\varepsilon_2}{2} - \left(\frac{\varepsilon_3}{2} + 1 - \frac{d_2}{p}\right)\bar{g}(\infty)\right] \|v_x(t)\|^2 + \frac{1}{2\varepsilon_2}\rho_1(t), \end{aligned} \tag{3.28}$$

for all $\delta_1 \in (0,1)$ and ε_2, $\varepsilon_3 > 0$, where $\rho_1(t) = \|F_1(t)\|^2 + \|F_2(t)\|^2$.

Proof. By multiplying (3.11) by $(u(x,t), v(x,t))$ and integrating over $(0,1)$, we get

$$\begin{aligned} \Psi'(t) =& -\bar{a}(t; u(t), u(t)) - \left\|\sqrt{\mu_2(t)} v_x(t)\right\|^2 + \int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle ds \\ &+ \langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle + \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle. \end{aligned} \tag{3.29}$$

On the other hand, we have

$$\begin{aligned} &\langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle \leq \frac{\varepsilon_2}{2}\left(\|u(t)\|_a^2 + \|v_x(t)\|^2\right) + \frac{1}{2\varepsilon_2}\rho_1(t), \\ &\int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle ds \leq \frac{1}{2\varepsilon_3}(g * v)(t) + \left(\frac{\varepsilon_3}{2} + 1\right)\bar{g}(t)\|v_x(t)\|^2, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \\ &\leq \frac{d_2}{p}\left[(g_* \diamondsuit u)(t) + (g * v)(t) + \bar{a}(t; u(t), u(t)) + \left\|\sqrt{\mu_2(t)} v_x(t)\right\|^2 - \bar{g}(t)\|v_x(t)\|^2\right] \\ &\quad - (1-\delta_1)\frac{d_2}{p}\eta^*\left(\|u(t)\|_a^2 + \|v_x(t)\|^2\right) - \frac{\delta_1 d_2}{p} I(t). \end{aligned}$$

Then, it follows from (3.29), (3.30) that (3.28) holds. Lemma 3.9 is proved. □

Based on the above results, we deduce the main result in this subsection as follows.

Theorem 3.10. *Assume that $(D_2) - (D_6)$ hold. Let $(\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1$ such that $I(0) > 0$ and the initial energy $E(0)$ satisfy (3.19). Then, there exist positive constants $\bar{C}$, $\bar{\gamma}$ such that*

$$\|u(t)\|_a^2 + \|v_x(t)\|^2 \leq \bar{C} \exp\left(-\bar{\gamma}\int_0^t \zeta(s) ds\right),\ \ \forall t \geq 0. \tag{3.31}$$

Proof. First, by the definition of $\mathcal{L}(t)$ and the inequalities (3.16), (3.28), we get

$$\begin{aligned} \mathcal{L}'(t) \leq& -\left(1 - \bar{\lambda}_* - \frac{\varepsilon_1}{2}\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - \tilde{\theta}_3(g_* \diamondsuit u)(t) \\ &+ \delta d_3(g * v)(t) - \frac{\delta\delta_1 d_2}{p} I(t) - \delta\tilde{\theta}_1\|u(t)\|_a^2 - \delta\tilde{\theta}_2\|v_x(t)\|^2 + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho_1(t), \end{aligned} \tag{3.32}$$

with

$$\begin{aligned}
d_3 &= \frac{1}{2\varepsilon_3} + \frac{d_2}{p}, \\
\tilde{\theta}_1 &= \tilde{\theta}_1(\delta_1, \varepsilon_2) = \left[(1-\delta_1)\frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{1*} - \frac{\varepsilon_2}{2}\right], \\
\tilde{\theta}_2 &= \tilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) = \left[(1-\delta_1)\frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{2*} - \frac{\varepsilon_2}{2} - \left(\frac{\varepsilon_3}{2} + 1 - \frac{d_2}{p}\right)\bar{g}(\infty)\right], \\
\tilde{\theta}_3 &= \bar{k}_* - \frac{\delta d_2}{p}.
\end{aligned} \tag{3.33}$$

By $p > d_2$ and $0 < \bar{g}(\infty) < \dfrac{d_2\eta^*}{p - d_2} + \mu_{2*}$, we also have

$$\begin{aligned}
\lim_{\delta_1 \to 0_+,\ \varepsilon_2 \to 0_+} \tilde{\theta}_1(\delta_1, \varepsilon_2) &= \frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{1*} > 0, \\
\lim_{\delta_1 \to 0_+,\ \varepsilon_2 \to 0_+,\ \varepsilon_3 \to 0_+} \tilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) &= \left[\frac{d_2}{p}\eta^* + \left(1 - \frac{d_2}{p}\right)\mu_{2*} - \left(1 - \frac{d_2}{p}\right)\bar{g}(\infty)\right] > 0.
\end{aligned} \tag{3.34}$$

Thus, we can choose $\delta_1 \in (0,1)$ and ε_2, $\varepsilon_3 > 0$ small enough such that

$$\tilde{\theta}_1 = \tilde{\theta}_1(\delta_1, \varepsilon_2) > 0,\ \ \tilde{\theta}_2 = \tilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) > 0. \tag{3.35}$$

With $1 - \bar{\lambda}_* > 0$, we also can choose $\delta > 0$ and $\varepsilon_1 > 0$ small enough such that

$$\tilde{\theta}_3 = \bar{k}_* - \frac{\delta d_2}{p} > 0,\ \ 1 - \bar{\lambda}_* - \frac{\varepsilon_1}{2} > 0. \tag{3.36}$$

Put

$$\theta_* = \min\left\{\delta\tilde{\theta}_1, \delta\tilde{\theta}_2, \tilde{\theta}_3, \frac{\delta\delta_1 d_2}{p}\right\}, \tag{3.37}$$

it implies from (3.32), (3.33), (3.36), (3.37) that

$$\mathcal{L}'(t) \le -\theta_* E_1(t) + (\theta_* + \delta d_3)(g * v)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\rho_1(t). \tag{3.38}$$

Combining (3.16) and (3.38), it leads to

$$\zeta(t)\mathcal{L}'(t) \le -\theta_*\zeta(t)E_1(t) - 2(\theta_* + \delta d_3)E'(t) + \bar{C}_1 e^{-\bar{\gamma}_0 t}, \tag{3.39}$$

with $\bar{C}_1 = \left[\dfrac{\theta_* + \delta d_3}{\varepsilon_1} + \dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\zeta(0)\right]\bar{C}_0$.

For convenience, we consider the new functional $L(t) = \zeta(t)\mathcal{L}(t) + 2(\theta_* + \delta d_3)E(t)$, then

$$L(t) \le \left[\zeta(0)\beta_2 + 2(\theta_* + \delta d_3)\bar{\beta}_2\right]E_1(t) \equiv \bar{\beta}_3 E_1(t), \tag{3.40}$$

and

$$L'(t) \le -\theta_*\zeta(t)E_1(t) + \bar{C}_1 e^{-\bar{\gamma}_0 t} \le -\frac{\theta_*}{\bar{\beta}_3}\zeta(t)L(t) + \bar{C}_1 e^{-\bar{\gamma}_0 t}. \tag{3.41}$$

Choosing $\bar{\gamma}$, $0 < \bar{\gamma} < \min\{\dfrac{\theta_*}{\bar{\beta}_3}, \dfrac{\bar{\gamma}_0}{\zeta(0)}\}$, from (3.41), we obtain

$$L'(t) + \bar{\gamma}\zeta(t)L(t) \le \bar{C}_1 e^{-\bar{\gamma}_0 t}. \tag{3.42}$$

Integrating (3.42), we deduce that

$$L(t) \le \left(L(0) + \frac{\bar{C}_1}{\bar{\gamma}_0 - \bar{\gamma}\zeta(0)}\right)\exp\left(-\bar{\gamma}\int_0^t \zeta(s)ds\right). \tag{3.43}$$

Hence, (3.27) and (3.43) lead to (3.31). Theorem 3.10 is proved completely. □

3.2.3 Blow-up and lifespan of the solution

This subsection is devoted to the study of the blow-up property for Prob. (3.11) when $F_1 = F_2 \equiv 0$.

First, we make the following assumptions

$(\tilde{H}_1)$ $(\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1$;

(B_2) $\mu_1, \mu_2 \in C^1([0,1] \times \mathbb{R}_+)$, and there exist the positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \geq \mu_{i*}, \mu_i'(x,t) \leq 0, \forall (x,t) \in [0,1] \times \mathbb{R}_+, i = 1,2$;

(B_3) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, such that

(i) $g(t) \geq 0, g'(t) \leq 0, \forall t \geq 0$,

(ii) $\mu_{2*} - \int_0^\infty g(s)ds > 0$,

(iii) $\int_0^\infty g(s)\,ds \leq \frac{p(p-2)\mu_{2*}}{(p-1)^2}$, with $\mu_* = \min\{\mu_{1*}, \mu_{2*}\}$;

(B_4) There exists the function $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and there exist the positive constants $d_1 > p$, such that

(i) $D_1\mathcal{F}(u,v) = f_1(u,v), D_2\mathcal{F}(u,v) = f_2(u,v), \forall (u,v) \in \mathbb{R}^2, i = 1,2$,

(ii) $uf_1(u,v) + vf_2(u,v) \geq d_1\mathcal{F}(u,v) \geq 0, \forall (u,v) \in \mathbb{R}^2$.

Remark 3.11. *The functions $f_1(u,v)$, $f_2(u,v)$ given in the example of Remark 3.5 also satisfy (B_4) (ii).*

Now, we consider the symmetric bilinear forms $\hat{a}_i(\cdot,\cdot)$, $(i = 1,2)$, defined by

$$\begin{aligned} \hat{a}_1(u,\varphi) &= \langle u,\varphi\rangle + \lambda_1 a(u,\varphi), \ (u,\varphi) \in V \times V, \\ \hat{a}_2(v,\psi) &= \langle v,\psi\rangle + \lambda_2\langle v_x,\psi_x\rangle, \ (v,\psi) \in H_0^1 \times H_0^1, \end{aligned} \tag{3.44}$$

and define the norms

$$\begin{aligned} \|u\|_{\hat{a}_1} &= \sqrt{\hat{a}_1(u,u)}, \ u \in V, \\ \|v\|_{\hat{a}_2} &= \sqrt{\hat{a}_2(v,v)}, \ v \in H_0^1, \end{aligned} \tag{3.45}$$

We can rewrite $\Psi(t)$ in (3.14) as follows

$$\Psi(t) = \frac{1}{2}\left(\|u(t)\|_{\hat{a}_1}^2 + \|v(t)\|_{\hat{a}_2}^2\right). \tag{3.46}$$

We also consider the functional $\bar{E}(t)$ defined by

$$\bar{E}(t) = \frac{1}{2}(g*v)(t) + \frac{1}{2}\bar{a}(t;u(t),u(t)) + \frac{1}{2}\left(\left\|\sqrt{\mu_2(t)}v_x(t)\right\|^2 - \bar{g}(t)\|v_x(t)\|^2\right) - \overline{\mathcal{F}}(t), \tag{3.47}$$

with

$$\overline{\mathcal{F}}(t) = \int_0^1 \mathcal{F}(u(x,t),v(x,t))\,dx. \tag{3.48}$$

We note more that $\bar{E}(t)$ is the functional $E(t)$ as in (3.13), with respect to $g_* \equiv 0$. Furthermore, $\bar{E}(0) = E(0)$.

Lemma 3.12. *Assume that $(\tilde{H}_1)$ and $(B_2) - (B_4)$ hold. Then we have*

$$\frac{d}{dt}\left[\bar{E}(t) + \int_0^t \left(\|u'(s)\|_{\hat{a}_1}^2 + \|v'(s)\|_{\hat{a}_2}^2\right)ds\right] \leq 0. \tag{3.49}$$

Moreover, the following energy inequality holds

$$\bar{E}(t) + \int_0^t \left(\|u'(s)\|_{\hat{a}_1}^2 + \|v'(s)\|_{\hat{a}_2}^2\right)ds \leq \bar{E}(0). \tag{3.50}$$

Proof. Multiplying the equation (3.11) by $(u'(x,t), v'(x,t))$ and integrating on $(0,1)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\bar{E}\left(t\right)+\int_0^t\left(\left\|u'(s)\right\|_{\hat{a}_1}^2+\left\|v'(s)\right\|_{\hat{a}_2}^2\right)ds\right] \\ &=\frac{1}{2}\left[\bar{a}'(t;u(t),u(t))+\int_0^1\mu_2'(x,t)v_x^2(x,t)dx\right]+\frac{1}{2}(g'*v)(t)-\frac{1}{2}\bar{g}(t)\left\|v_x(t)\right\|^2\leq 0,\end{aligned} \tag{3.51}$$

for any regular solution (u,v). We can extend (3.51) to weak solutions by using density arguments. Combining $(\tilde{H}_1)$ and $(B_2)-(B_4)$, Lemma 3.12 is proved. □

Theorem 3.13. *Assume that $(B_2)-(B_4)$ hold. Then, for any initial conditions $(\tilde{u}_0,\tilde{v}_0)\in V\times H_0^1$ such that $E\left(0\right)<0$, the weak solution of the Prob. (3.11) with respect to $F_1=F_2\equiv 0$ blows up at finite time and the lifespan T_∞ of the solution (u,v) satisfies*

$$T_\infty\leq\frac{-8(p-1)\Psi\left(0\right)}{p(p-2)^2\bar{E}\left(0\right)}\equiv T_\infty^{\max}. \tag{3.52}$$

Furthermore, if in addition the following assumptions

(B_{4*}) *There exists the constant $d_2>p$ such that*

(i) $uf_1(u,v)+vf_2(u,v)\leq d_2\mathcal{F}(u,v),\ \forall\,(u,v)\in\mathbb{R}^2,\ \forall\,(u,v)\in\mathbb{R}^2,$

(ii) $\mathcal{F}(u,v)\leq\bar{d}_2\sum_{i=1}^N\left(|u|^{q_i}+|v|^{\bar{q}_i}\right),\ \forall\,(u,v)\in\mathbb{R}^2,\ \forall\,(u,v)\in\mathbb{R}^2;$

(B_{5*}) $\displaystyle\int_{\Psi(0)}^\infty\frac{dz}{G_1\left(z\right)}\leq\frac{-8(p-1)\Psi\left(0\right)}{p(p-2)^2\bar{E}\left(0\right)},$

in which

$$\begin{aligned}&G_1\left(z\right)=\frac{4\bar{g}(\infty)}{\lambda_*}z+(1+d_2)d_3\sum_{i=1}^N\left(z^{q_i/2}+z^{\bar{q}_i/2}\right), \\ &\lambda_*=\min\{\lambda_1,\lambda_2\},\ d_3=\bar{d}_2\max\left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2},\left(\frac{2}{\lambda_*}\right)^{\bar{q}_i/2},\ i=1,\cdots,N\right\},\end{aligned} \tag{3.53}$$

then, the blow-up time T_∞ satisfies

$$T_\infty\geq\int_{\Psi(0)}^\infty\frac{dz}{G_1\left(z\right)}\equiv T_\infty^{\min}. \tag{3.54}$$

Remark 3.14. *The assumption (B_{5*}) holds if $\lambda_*=\min\{\lambda_1,\lambda_2\}$ is small enough.*

Proof. We first prove that the solution (u,v) obtained here is not a global solution in $\mathbb{R}_+$. Indeed, by contradiction, we will assume that the weak solution exists in the whole interval $\mathbb{R}_+$

For $\bar{E}(0)<0$, let $0<\beta\leq\frac{-p\bar{E}(0)}{p-1}$, $\tau>\frac{2\Psi\left(0\right)}{\beta(p-2)}$, and $T_0\geq\frac{\beta\tau^2}{(p-2)\beta\tau-2\Psi\left(0\right)}$, we define the auxiliary functional $\Gamma:[0,T_0]\longrightarrow\mathbb{R}$ as follows

$$\Gamma\left(t\right)=2\int_0^t\Psi\left(s\right)ds+2\left(T_0-t\right)\Psi\left(0\right)+\beta(t+\tau)^2,\ 0\leq t\leq T_0. \tag{3.55}$$

By direct computation, we have

$$\begin{aligned}\Gamma'\left(t\right)&=2\Psi\left(t\right)-2\Psi\left(0\right)+2\beta\left(t+\tau\right)=2\int_0^t\Psi'(s)\,ds+2\beta\left(t+\tau\right) \\ &=2\int_0^t\hat{a}_1(u'\left(s\right),u\left(s\right))ds+2\int_0^t\hat{a}_2(v'\left(s\right),v\left(s\right))ds+2\beta\left(t+\tau\right),\end{aligned} \tag{3.56}$$

and

$$\Gamma''(t) = 2\Psi'(t) + 2\beta. \tag{3.57}$$

Because of (3.55) and (3.56), it implies that $\Gamma(t) > 0$ for all $t \in [0, T_0]$ and $\Gamma'(0) = 2\beta\tau > 0$. Using the Cauchy-Schwarz, it follows from (3.56) that

$$\begin{aligned} \frac{1}{2}\left|\Gamma'(t)\right| &\le \left[\int_0^t \left|\hat{a}_1(u'(s), u(s))\right| ds + \int_0^t \left|\hat{a}_2(v'(s), v(s))\right| ds + \beta(t+\tau)\right] \\ &\le \sqrt{\sigma_1(t)}\sqrt{\sigma_2(t)} = \sqrt{\sigma(t)}, \end{aligned} \tag{3.58}$$

in which

$$\begin{aligned} \sigma(t) &= \sigma_1(t)\,\sigma_2(t), \\ \sigma_1(t) &= \int_0^t \left(\|u'(s)\|_{\hat{a}_1}^2 + \|v'(s)\|_{\hat{a}_2}^2\right) ds + \beta, \\ \sigma_2(t) &= \int_0^t \left(\|u(s)\|_{\hat{a}_1}^2 + \|v(s)\|_{\hat{a}_2}^2\right) ds + \beta(t+\tau)^2 = 2\int_0^t \Psi(s)\,ds + \beta(t+\tau)^2, \end{aligned} \tag{3.59}$$

then, (3.58) leads to

$$\sigma(t) \ge \frac{1}{4}\left|\Gamma'(t)\right|^2, \ \forall t \in [0, T_0]. \tag{3.60}$$

Therefore, since $\Gamma(t) = \sigma_2(t) + 2(T_0 - t)\Psi(0) \ge \sigma_2(t)$, we get

$$2p\Gamma(t)\,\sigma_1(t) \ge 2p\sigma_2(t)\,\sigma_1(t) = 2p\sigma(t) \ge \frac{p}{2}\left|\Gamma'(t)\right|^2. \tag{3.61}$$

From (3.61), it gives

$$\Gamma''(t)\,\Gamma(t) - \frac{p}{2}\left|\Gamma'(t)\right|^2 \ge 2\Gamma(t)\left[\frac{1}{2}\Gamma''(t) - p\sigma_1(t)\right] = 2\Gamma(t)\,D(t), \tag{3.62}$$

with

$$D(t) = \frac{1}{2}\Gamma''(t) - p\sigma_1(t). \tag{3.63}$$

On the other hand, by multiplying the equation in (3.11) by $(u(x,t), v(x,t))$, and then integrating over $(0,1)$, it follows from (3.57) that

$$D(t) = \beta + \Psi'(t) - p\sigma_1(t) \tag{3.64}$$

$$\begin{aligned} &= \beta - \bar{a}(t; u(t), u(t)) - \left\|\sqrt{\mu_2(t)}v_x(t)\right\|^2 + \int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle ds \\ &\quad + \langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \\ &\quad - p\left[\int_0^t \left(\|u'(s)\|_{\hat{a}_1}^2 + \|v'(s)\|_{\hat{a}_2}^2\right) ds + \beta\right]. \end{aligned} \tag{3.65}$$

We can estimate terms on the right hand side of (3.64)

$$\int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle ds \ge -\frac{p}{2}(g * v)(t) + \left(1 - \frac{1}{2p}\right)\left(\bar{g}(t)\,\|v_x(t)\|^2\right), \tag{3.66}$$

and

$$\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \ge d_1\overline{\mathcal{F}}(t). \tag{3.67}$$

Combining (3.47), (3.50), (3.64)-(3.67), we obtain

$$\begin{aligned} D\left(t\right) \geq & -p\bar{E}\left(0\right) - (p-1)\beta + \left(d_1 - p\right)\overline{\mathcal{F}}(t) + \frac{(p-2)\mu_{1*}}{2}\left\|u(t)\right\|_a^2 \\ & + \frac{(p-1)^2}{2p}\left[\frac{p(p-2)\mu_{2*}}{(p-1)^2} - \bar{g}(\infty)\right]\left\|v_x(t)\right\|^2 \ \geq 0, \end{aligned} \tag{3.68}$$

because of $d_1 > p > 2$, $0 < \bar{g}(\infty) \leq \dfrac{p(p-2)\mu_{2*}}{(p-1)^2}$ and $0 < \beta \leq \dfrac{-p\bar{E}(0)}{p-1}$.

It implies from (3.62) and (3.68) that

$$\Gamma^{\frac{p}{2}-1}\left(t\right) \geq \frac{2\Gamma^{\frac{p}{2}}\left(0\right)}{(p-2)\Gamma'\left(0\right)}\frac{1}{T_* - t}, \ \forall t \in [0, t_{\min}), \tag{3.69}$$

where $t_{\min} = \min\{T_*, T_0,\}$, with $T_* = \dfrac{2\Gamma\left(0\right)}{(p-2)\,\Gamma'\left(0\right)}$.

By $0 < \beta \leq \dfrac{-p\bar{E}(0)}{p-1}$, $\tau > \dfrac{2\Psi\left(0\right)}{\beta(p-2)}$ and $T_0 \geq \dfrac{\beta\tau^2}{(p-2)\beta\tau - 2\Psi\left(0\right)}$, we get

$$T_* = \frac{2\Gamma\left(0\right)}{(p-2)\,\Gamma'\left(0\right)} = \frac{2T_0\Psi\left(0\right) + \beta\tau^2}{(p-2)\,\beta\tau} \in \left(0, T_0\right]. \tag{3.70}$$

Because (3.69), it gives $\lim\limits_{t\to T_*^-}\Gamma\left(t\right) = +\infty$. This is a contradiction.

Consequently, the solution (u, v) blows up at finite time.

Now, we find a upper bound for T_∞.

It is clear to see that $T_\infty \leq \dfrac{2T_\infty\Psi\left(0\right) + \beta\tau^2}{(p-2)\,\beta\tau}$, it is equivalent to

$$T_\infty \leq \frac{\beta\tau^2}{(p-2)\,\beta\tau - 2\Psi\left(0\right)}, \ \forall\left(\beta, \tau\right) \in \tilde{D}\left(\tilde{u}_0, \tilde{v}_0\right), \tag{3.71}$$

in which

$$\tilde{D}\left(\tilde{u}_0, \tilde{v}_0\right) = \left\{(\beta, \tau) \in \mathbb{R}_+^2 : 0 < \beta \leq \frac{-p\bar{E}\left(0\right)}{p-1}, \ \tau > \frac{2\Psi\left(0\right)}{\beta(p-2)}\right\}. \tag{3.72}$$

Consider the function $H(\tau, \beta) = \dfrac{\beta\tau^2}{(p-2)\,\beta\tau - 2\Psi\left(0\right)} = \dfrac{\tau^2}{(p-2)(\tau - \tau_*)}$, $(\beta, \tau) \in \tilde{D}\left(\tilde{u}_0, \tilde{v}_0\right)$, with $\tau_* = \dfrac{2\Psi\left(0\right)}{\beta(p-2)}$.

Fixed β, $0 < \beta \leq \dfrac{-p\bar{E}\left(0\right)}{p-1}$. We have $\dfrac{\partial H}{\partial\tau}(\tau, \beta) = \dfrac{\tau\left(\tau - 2\tau_*\right)}{(p-2)\left(\tau - \tau_*\right)^2}$, $\forall\tau > \tau_*$, this implies that the function $\tau \longmapsto H(\tau, \beta)$ is decreasing in $\left(\tau_*, 2\tau_*\right)$, and increasing in $\left(2\tau_*, +\infty\right)$, so

$$\begin{aligned} H(\tau, \beta) \geq H(2\tau_*, \beta) &= \frac{4\tau_*}{p-2} = \frac{8\Psi\left(0\right)}{\beta(p-2)^2} \\ &\geq \frac{8\Psi\left(0\right)}{\dfrac{-p\bar{E}\left(0\right)}{p-1}(p-2)^2} = \frac{-8(p-1)\Psi\left(0\right)}{p(p-2)^2\bar{E}(0)} = T_\infty^{\max}, \ \forall(\beta, \tau) \in \tilde{D}\left(\tilde{u}_0, \tilde{v}_0\right). \end{aligned} \tag{3.73}$$

From (3.71) and (3.73), it leads to $T_\infty \leq \dfrac{-8(p-1)\Psi\left(0\right)}{p(p-2)^2\bar{E}(0)} = T_\infty^{\max}$. Hence, (3.52) holds.

Next, we seek a lower bound for the blow-up time T_∞. We have

$$\Psi'(t) = -\bar{a}(t;u(t),u(t)) - \left\|\sqrt{\mu_2(t)}v_x(t)\right\|^2 + \int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle ds \quad (3.74)$$
$$+\langle f_1(u(t),v(t)),u(t)\rangle + \langle f_2(u(t),v(t)),v(t)\rangle.$$

The terms on the right hand side of (3.74) are also estimated as follows

$$\int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle ds \le \frac{1}{2}(g*v)(t) + \frac{3}{2}\bar{g}(t)\left\|v_x(t)\right\|^2, \quad (3.75)$$

$$\langle f_1(u(t),v(t)),u(t)\rangle + \langle f_2(u(t),v(t)),v(t)\rangle \le d_2\overline{\mathcal{F}}(t), \quad (3.76)$$

$$\|u(t)\|_a^2 + \|v_x(t)\|^2 \le \frac{2}{\lambda_*}\Psi(t), \text{ with } \lambda_* = \min\{\lambda_1,\lambda_2\}, \quad (3.77)$$

$$\overline{\mathcal{F}}(t) \le \bar{d}_2\sum_{i=1}^N \left(\|u(t)\|_{L^{q_i}}^{q_i} + \|v(t)\|_{L^{\bar{q}_i}}^{\bar{q}_i}\right) \le d_3\sum_{i=1}^N \left[(\Psi(t))^{q_i/2} + (\Psi(t))^{\bar{q}_i/2}\right], \quad (3.78)$$

in which $d_3 = \bar{d}_2 \max\left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\bar{q}_i/2}, \ i=1,\cdots,N\right\}$.

On the other hand

$$\bar{E}(t) + \overline{\mathcal{F}}(t) - \frac{1}{2}\bar{a}(t;u(t),u(t)) - \frac{1}{2}\left(\left\|\sqrt{\mu_2(t)}v_x(t)\right\|^2 - \bar{g}(t)\left\|v_x(t)\right\|^2\right) = \frac{1}{2}(g*v)(t). \quad (3.79)$$

Combining (3.50), (3.74)-(3.80), it gives

$$\Psi'(t) \le G_1(\Psi(t)), \quad (3.80)$$

where $G_1(z)$ is defined as in (3.53).

By (3.80), it leads to

$$t \ge \int_0^t \frac{\Psi'(s)ds}{G_1(\Psi(s))} = \int_{\Psi(0)}^{\Psi(t)} \frac{dz}{G_1(z)}. \quad (3.81)$$

Therefore, we derive the lower bound for T_∞ as follows

$$T_\infty \ge \int_{\Psi(0)}^\infty \frac{dz}{G_1(z)} = T_\infty^{\min}. \quad (3.82)$$

Theorem 3.13 is proved completely. □

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